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ON THE ELECTRICAL RESISTANCE OF THIN FILMS.

By Isabelle Stone.

THE discrepancy between the resistance of a film of given thickness measured directly by the Wheatstone bridge, as compared with the resistance calculated from its weight, density and dimensions, suggested the present investigation. As shown farther on by tables and curves, the calculated resistance is but a small fraction of the initial resistance of the film, when measured directly. The process of experimenting, however, revealed the fact that the initial resistance was not constant, but decreased with time, reaching its lowest value only after an infinite time. Films have not been preserved for longer intervals than three or four months; but the values of the resistance at the end of this length of time, while only approximately constant, nevertheless compare very favorably with the calculated values for the same thickness. This paper makes no pretensions of giving an exhaustive account of the subject under consideration. Many lines of research are only touched upon; none are worked out fully. The only metal studied is silver. The other metals, as well as the entire subject of films deposited in a vacuum, have not yet been touched upon. Investigation in all these directions, which is needed in order to have a good conception of the action of films, will be taken up at a future time.

As before stated, the metal chosen for experiment was silver deposited by the "Rochelle Salt Process." The silver was deposited on glass plates which were weighed before and after the silvering. From the weight of the silver and the dimensions of the glass plate,

the surface density was calculated. The glass plates were 8×0.8 centimetres in dimensions and were laid face downward in such a way that the ends rested on a packing of silver leaf over the copper terminals. Good contact between the glass and terminals was insured by covers of fibre screwed down above and below the glass plate. Too great pressure, which would break the glass or tear the film, was avoided by cushions of paper at either end of the covers.

It was at once noticed that the resistance of the same film was not constant, but varied from hour to hour. Consecutive readings showed a steady decrease in resistance, the fall being more marked the higher the original resistance. The following table illustrates this fact.

TABLE I.

Film.	Weight.	R_0	Successive Readings of R directly after silvering.	R after 24 hours.
(a)	.24 mg.	1372	1261, 1239, 1166, 1097	550, 543, 530
(b)	.27 mg.	628	480, 338, 290	203
(c)	.36 mg.	93	78, 77.5, 77	44.1, 43, 42.4
(d)	1.44 mg.	5.9	5.9	5.6

Thinking that this decrease might be due to moisture, the films were kept in dessicators except when taking measurements, but the same regular fall of resistance with time was observed.

After it was discovered that the resistance of a film at any time was a function of the age of the film, an attempt was made to find out the rate of fall. Readings of the resistance of a new film were taken at intervals, fifteen minutes apart, for several hours, and consecutive readings taken on the same film after an interval of sixteen or twenty-four hours. A film whose initial resistance was 528 ohms, fell in 4½ hours to 404 ohms. The first half of the time the average fall was 6 ohms every 15 minutes, and the last half 4 ohms in the same interval. After 16 hours the resistance of the film was 225 ohms, and in the next 30 minutes the fall was 3 ohms. It was found that the greatest fall in resistance occurred in the first 24 hours. Also, that the higher the original resistance of the film, the greater the fall in the first 24 hours. The curves show this very clearly.

Experiments were tried to show the effect of heat on the films. It was found that heat accelerated the time effect. To show this, the comparison of a heated and unheated film is annexed.

TABLE II.

		R_0	After 5 minutes heat.	After 5 hours heat.	After 24 hours.
Thick film.	heated.	(a) 130	8.9	4.8	4.9
	not heated.	(b) 88	72	50	38
Thin film.	heated.	35566	223	87	89
	not heated.	35000	35000	31252	14614

Two similar films (*a*) and (*b*) were taken ; (*a*) was heated for five minutes and (*b*) not heated. A comparison of the resistances of both (*a*) and (*b*) was made after (*a*) was removed from the oven ; (*a*) was then heated for five hours, and another comparison between resistances made. A thin film shows the fall of resistance as effected by heat in an even more marked degree.

To find the effect of moisture on a film, a film was heated in an oven which also contained a vessel of boiling water. A comparison unheated film was made and its fall of resistance compared with that of the heated film.

TABLE III.

		R_0	After 5 minutes heat.	After 5 hours heat.	24 hours after silvering.
Thick film.	heated with moisture.	193	10	5.4	6
	not heated.	90	—	52	40
Thin film.	heated with moisture.	3380	254	35500	—
	not heated.	2164	—	761	—

(It is to be noticed that the thin film failed to hold after 5 hours of heating.) Any differences between a film heated with dry heat or in the presence of moisture could not be detected.

A series of experiments was undertaken to find how much heating was necessary to lower the resistance of a film to its minimum value, and if less than this period of heating was allowed, to see whether the film continued to fall with time or not. Also, at what point in the curve illustrating the fall of resistance with time, the heat when applied would have the greatest effect in lowering the resistance. The experiments were conducted according to the following methods :

Comparison unheated films were kept for each heated film.

A. (*a*) The film was not disturbed for one hour after silvering. Readings of its resistance having been taken initially and at the end of the hour it was put in the oven and heated one minute, removed, measured and left undisturbed afterward.

(*b*) A new film was treated similarly, except that it was heated for 2 minutes.

(*c*) A new film was treated similarly, except that it was heated for 3 minutes, etc.

B. (*a*) After measurement the film was not disturbed for 5 hours. It was then measured, heated one minute, measured again and put aside.

(*b*) A new film was left undisturbed for 5 hours, and heated 2 minutes.

(*c*) A new film was left undisturbed for 5 hours and heated 3 minutes, etc.

C. (*a*) After measurement the film was not disturbed for 16 hours. It was then measured, heated one minute, measured again and put aside.

(*b*) A new film was left undisturbed for 16 hours and heated 2 minutes, etc.

Each of the above series of experiments was carried out for a thick film and also for a thin film.

By consulting these tables, it can be seen that

(1) Heat accelerates the time effect.

(2) The greater the age of the film, the less the effect produced on it by heat.

(3) In general, the longer the heat is applied, the greater the fall in the resistance.

TABLE IV.

		R_0	After 1 hour.	Heated ½ min.	Fall in R.	After 48 hrs.	R_0	After 1 hr.	Heated 1 min.	Fall in R.	After 48 hrs.	R_0	After 1 hr.	Heated 2 min.	Fall.	After 48 hrs.
Thick film.	heated.	22	19.8	16.4	3.4	12	42.2	37.7	19.9	17.8	10.8	18	16	7	9	7.2
	not heated.	14.2	12.1	11	1.1	9	56.4	48	46	2	14	15	13	13	0	11
Thin film.	heated.	1616	1488	1102	386	503	2542	2116	813	1303	420	1018	920	190	730	134
	not heated.	2088	2088	1987	101	527	914	882	800	82	538	6089	3746	—	—	734

TABLE V.

		R_0	After 1 hour.	Heated 3 min.	Fall.	After 48 hrs.	R_0	After 1 hr.	Heated 4 min.	Fall.	After 48 hrs.	R_0	After 1 hour.	Heated 5 min.	Fall.	After 53 hrs.
Thick film.	heated.	15.5	13.4	7	6.4	7.2	13.4	13.1	7.2	5.9	8.1					
	not heated.	15	14	—	—	12.6	13.6	13	—	—	12					
Thin film.	heated	1204	1014	133	881	94	1015	939	170	769	128	2390	1456	336	1120	271
	not heated.	567	482	474	8	65	266	251	—	—	111	510	347	—	—	126

TABLE VI.

		R_0	After 1 hour	Heated 6 min.	Fall	After 1 hour	R_0	After 1 hour.	Heated 7 min.	Fall	After 1 hour	R_0	After 1 hour	Heated 8 min.	Fall	After 1 hour
Thin film.	heated	1155	780	47.5	732.5	42	1933	1262	101.8	1160.2	94.7	3413	2831	86.4	2744.6	84.2

TABLE VII.

		R_0	After 5 hrs.	Heated 3 Min.	Fall	After 53 hrs.	R_0	After 5 hrs.	Heated 4 min.	Fall		R_0	After 5 hrs.	Heated 5 min.	Fall	After 24 hrs.
Thick film.	heated						53	40	35	5		61	27	11.5	15.5	9.7
Thin film.	heated	1293	624	470	154	462										

TABLE VIII.

		R_0	After 5 hrs.	Heated 6 min.	Fall		R_0	After 5 hrs.	Heated 7 min.	Fall		R_0	After 16 hrs.	Heated 5 min.	Fall	After 10 hrs.
Thick film.		92	46.8	15.9	30.9		155	80.5	14.7	65.8		30	22.7	21.7	1	17.9
Thin film.												1135	128	61.7	66.3	58

TABLE IX. (Thick film.)

R_0	After 16 hrs.	Heated 6 min.	Fall	After 5 hrs.	R_0	After 16 hrs.	Heated 7 min.	Fall	After 5 hrs.
34.5	16.4	8.4	8	8.3	33.4	16.7	8.7	8	8.9

(4) A certain period of heating reduces the resistance of each film to its lowest value.

(5) This period of heating, necessary to produce the minimum resistance, is greater, the greater the initial value of the film.

(6) After the minimum value of the resistance is reached, further heating has no effect upon the film.

(7) When not reduced to its minimum resistance by heat, the film continues to decrease with time in the ordinary manner.

(8) When the lowest value of the resistance is produced by heating, there seems to be a certain increase in the resistance, when measured after a number of hours.

To find whether heat produced any effect on comparatively thick films, a plate was placed in the silver bath three or four times and a heavy coating of silver thereby secured. The initial resistance was 5 ohms. After one half hour, it had fallen to 4.9 ohms. On heating it for 5 minutes the resistance fell to .98 and when measured after a rest of 72 hours, its resistance was .97 ohms. It would thus seem to be true that heating produces an effect upon films of all thicknesses.

Experiments were tried to see if an electric current would have the same effect as heat upon a film. Many experiments were tried to determine the proper current which would have an effect upon the film without destroying it. The results roughly obtained were not entirely satisfactory. The fall in resistance of a film as an effect of current was in excess of the corresponding fall as an effect of time in the comparison film; but it was not proved that this excess of fall was due to the electric current and not to the heating effect of the current. It was also noticed that with films of high resistance, the short currents sent through the film, while a bridge balance was being obtained, lowered the resistance noticeably. The question as to whether this change was due to heat or electric current demands a separate investigation.

In taking consecutive readings of the resistance of a film, it was noticed that the slamming of a door or other violent vibration produced a small but instantaneous decrease in the resistance. A few experiments were made by placing a film on the resonator of a tuning fork in vibration and measuring its resistance before and after, also by striking the table on which the film lay. The results seemed to confirm the fact that shocks hastened the fall of resistance; but the experiments were not carried further.

It was decided to take films varying in initial resistance from 5 to 100,000 ohms, observe the resistance from day to day and construct curves for each film, taking times as abscissæ and resistances as ordinates.

A and *B* are the bottom and top clamp of fibre, screwed together by screws through the four holes; *d*, *e*, *f*, *g* are cushions of tissue paper; *m* and *n* are copper terminals over which silver leaf is

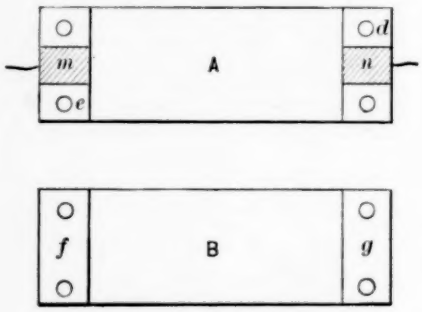

Fig. 1.

placed. The glass plate is the length of the clamp and the width of the terminals. It rests silvered face downward, upon the silver leaf. Twelve covers of hard rubber were made for the twelve films to be experimented upon. Sheet rubber was used as an insulator, and to prevent the breaking of the glass by pressure from the hard rubber covers. A sulphide was formed on the silver, due to the sulphur from the rubber. The effect was much more noticeable in those films which were heated. When kept in a dessicator, the sulphide continued to form on the films, though not so rapidly as when they were kept in the air. The presence of the sulphide manifested itself sooner on the thin than on the thick films by an

increase in the resistance. The rubber covers and cushions were, therefore discarded. Fibre covers were tried with cushions of tissue paper. A set of curves recorded in the tables as films set up between December 19, 1896, and January 7, 1897, were the result of a fairly good set of readings. The effect of the sulphide in increasing the resistance did not manifest itself except on the two films of highest initial resistance and even in those cases not till after a month had expired.[1] It was noticed that temperature had an appreciable effect on the resistance, the effect of cold being to retard and of heat to accelerate the decrease in the resistance.

A new series of experiments was undertaken, aiming to carry out three precautions.

(1) To avoid a sulphide formation.

(2) To keep the film at a constant temperature.

(3) To use the same silvering solution for all the films.

Having found that sulphide was formed in the presence of air it was decided to isolate the films as much as possible. Each terminal containing the film to be examined was placed in a glass jar, known to contain air as pure as possible. A tin cover was screwed down upon the top of the jar, the only openings being two small holes to allow the wires passage. These jars, twelve in number, were placed in a box completely enclosed and lined with felt. The box was placed in a constant temperature room and removed for a few minutes only each day in order to take the necessary measurements and note any changes in temperature given by a thermometer placed in the box. The three series of curves for the films set up on March 2, March 4 and March 11, show the effect of these conditions.

The reason for setting up three distinct series of films from the same solution was to try to ascertain whether the age of the solution had an effect upon the fall of resistance. Observations were made on films of initially the same resistance, set up from the same solution; but at intervals of several days apart. By comparison of the resistance readings, it was concluded that the age of the solution had to do with the fall of resistance, the fall in the first forty-

[1] Due probably to escape of illuminating gas.

eight hours being much less the older the solution. An analogous phenomenon was observed in the heat experiments. However, films of less than 100 ohms initial resistance did not show this characteristic.

In order to prevent completely the formation of the sulphide on the silver film, the terminal, enclosing the film to be observed, was placed in a glass tube, connected with a mercury air pump, the air exhausted and the tube sealed off. The results are given in tables under films set up on March 26, 27 and 30.

Under these circumstances a film of initially very high resistance, 1.4,000 ohms, decreased for a few days in resistance to 80,000 ohms and then increased indefinitely. As this could not now be due to the formation of the sulphide, it was concluded that the preservation of films of an initial resistance above 50,000 ohms was impossible with the methods of experimentation hitherto used.

TABLE X.

First Series. Films kept in dessicator.

Curves.	Date.	Solution.	C_1	C_2	$C_1 + C_2$	n	k	Temp.	Weight in mg.	Time.
(d)	Dec. 19.	B	291	27280	27571	.43	.52		.12	53 days
(m)	Jan. 7.	C	386	5403	5789	.43	.40		.22	34 days
(g)	Dec. 21.	B	87	1073	1160	.43	.28		.38	51 days
(b)	Dec. 22.	B	26	160	186	.43	.19		.62	50 days
(j)	Dec. 22.	B	8.4	28.6	37	.43	.18		1.52	48 days
(k)	Dec. 23.	B	4.3	5.2	9.5	.43	.12		1.61	49 days
I	Nov. 7.	D	84	1202	1286	.43	.30			17 days
II	Nov. 5.	D	16	512	528	.43	.34			27 days

$$y = c_1 + c_2 e^{-kt^n}$$

$c_1 = y$ when $t = \infty$.

$c_1 + c_2 = y$ when $t = 0$.

Time means period during which observations were taken.

TABLE XI.

Second Series. Films in glass cans kept in felt-lined boxes.

Curves.	Date.	Solution.	C_1	C_2	$C_1 + C_2$	n	k	Temp.	Weight.	Time in days.
(d)	Mch. 2.	A.	992	31538	32530	.43	.85	21°		48
(m)	"	"	416	18984	19400	.43	.80	"		"
(i)	"	"	260	7140	7400	.43	.75	"		"
(g)	"	"	123	2447	2570	.43	.60	"		"
(h)	"	"	24	408	432	.43	.44	"		"
(c)	"	"	19.2	11.97	31.13	.43	.19	"		"
(k)	"	"	5.6	12.14	17.7	.43	.18	"		"
(b)	Mch. 4.	"	1821	42379	44200	.43	.76	20.5		40
(f)	"	"	600	13940	14540	.43	.60	"		"
(a)	"	"	415	6185	6600	.43	.37	"		"
(j)	"	"	156	1894	2050	.43	.30	"		"
(e)	"	"	41	711	752	.43	.55	"		"
O	Mch. 11.	"	938	31162	32100	.43	.51	19°3		65
R	"	"	642	25995	26098	.43	.43	"		"
L	Mch. 10.	"	370	9730	10100	.43	.42	"		"
P	Mch. 11.	"	320	4550	4870	.43	.52	"		"
T	"	"	190	3384	3574	.43	.43	"		"
Q	"	"	190	2170	2360	.43	.34	"		"
S	"	"	159	651	810	.43	.36	"		"

TABLE XII.

Third Series. Films kept in vacuum tubes.

Curves.	Date.	Solution.	C_1	C_2	$C_1 + C_2$	n	k	Temp.	Wgt.	Time in days.
(9)	Mch. 27.	F	1600	42800	44400	.43	.25	19.°3		50
(7)	"	"	19090	5510	24600	.43	.11	"		"
(10)	"	"	450	15550	16000	.43	.26	"		"
(6)	Mch. 26.	"	233	4677	4910	.43	.23	"		"
(8)	"	"	90	954	1044	.43	.25	"		"
(2)	Mch. 27.	"	80	710	790	.43	.23	"		"
(5)	Mch. 30.	E	869	9001	9870	.43	.23	19.°3		"
(11)	"	"	2248	3052	5300	.43	(.68)	"		"
(3)	"	"	106	1324	1430	.43	.44	"		"
(1)	"	"	85	196	281	.43	.28	"		"

(11) increased in resistance after the fourth day. The value of k therefore could be obtained only approximately. From an analogy with (5) it was concluded that the value of k should be nearer .5. The fall of (7) was noticeably small.

The connection between the initial resistance of a film and its thickness is shown in the following table:

TABLE XIII.

	Thickness in centimeters.	R_0 Resistance of a film 10×1 cm.	R Calculated resistance.	$\frac{R_0}{R}$ Ratio of experimental to calc. R.	Thickness in centimeters.	R_0 Resistance of a film 10×1 cm.	R Calculated resistance.	$\frac{R_0}{R}$
Solution A. Dates, June 21 and June 22.	.0000013	435284	8	54410	.00000096	1188470	11.3	105174
	.0000019	13733	5.6	2452	.0000021	15176	4.7	3229
	.0000032	3545	3.1	1143	.0000019	12765	5.4	2364
	.0000036	1043	3.0	348	.0000029	1356	3.7	367
	.0000085	198	1.2	165	.0000108	170	1.0	170
	.0000088	194	1.1	161	.0000157	67	.6	111
	.0000192	33	.5	66				
	.0000224	19	.4	47				
Solution B. Dates, June 25 and June 26.	.0000035	10157	2.8	3627	.0000019	141833	5.4	26265
	.0000037	4423	2.9	1525	.0000030	13937	3.4	4037
	.0000045	1010	2.2	463	.0000037	2935	2.7	1083
	.0000083	344	1.2	286	.0000041	1173	2.4	488
	.0000156	132	.6	220	.0000056	537	1.8	301
	.0000173	47	.5	94	.0000081	234	1.2	195
					.0000239	55	.4	137

TABLE XIV.

	Thickness in centimeters.	R_0	R.	$\frac{R_0}{R}$
Solution C. Date, June 29.	.0000026	260855	4.2	62108
	.0000037	16215	2.8	5836
	.0000054	2744	1.9	1506
	.0000075	260	1.5	178
	.0000123	127	.9	142

Density of silver, 10.53.

The second column gives the experimental value of the resistance of a film of known thickness, ten centimeters in length and one centimeter in width. The calculated resistance is found by dividing the specific resistance in ohms times the length by the thick-

ness times the width. The thickness was calculated from the weight of the film by dividing the weight in grams by the area times the density. Curves were drawn taking the thicknesses, estimated in millionths of a centimeter, as abscissæ and the ratio $\frac{R_0}{R}$ as ordinates.

The calculated resistance for a given thickness is but a small fraction of the experimental value for the same thickness.

Three different solutions were used in this set of experiments. It can be seen from the curves that two series of weighings made on two successive days ; but from the same solution agree closely, while there is a difference in the form of the curve for different solutions.

Having drawn the curves given by the experimental values of the film at times when observations were taken, the theoretical curve, to which each experimental curve most closely corresponds, had to be deduced. The theoretical curve given by $y = C_1 + C_2 e^{-kt^n}$ corresponds most closely with the experimental curves. The tables give the calculated values for the constants C_1, C_2 and k, taking n throughout the same for each curve. The agreement between the experimental and theoretical curves was very close.

From an observation of the curves and the values of k corresponding to the curves it is apparent that the lower the value of k the slower the fall of resistance. Hence it can be seen at a glance, in the second series, by comparison of the values of k, that the fresher the solution the more rapid the fall. For any one of the three sets of curves in this series, the decrease in k as the initial resistance decreases follows approximately the experimental law $r = ce^{km}$ where values of $c_1 + c_2$ are taken as abscissæ and corresponding values of k as ordinates, the constants being c and m. For the curves drawn from films kept in a vacuum, a somewhat different order holds. For each curve, no matter what the initial value of the resistance, k is approximately the same, *i. e.*, the law $y = c_1 + c_2 e^{-kt^n}$ is very closely followed as $y = 42 + 958e^{-.25t^{.43}}$. On looking at the tables, it is seen that the curves for the vacuum films are distinguished from the others by the slowness with which they fall. The higher the value of k the more rapidly the resistance falls. For vacuum films k is approximately .25, whereas, in other cases, the value varies from .85 to .18. Curve (7) $R_0 = 24600$ exhibits a

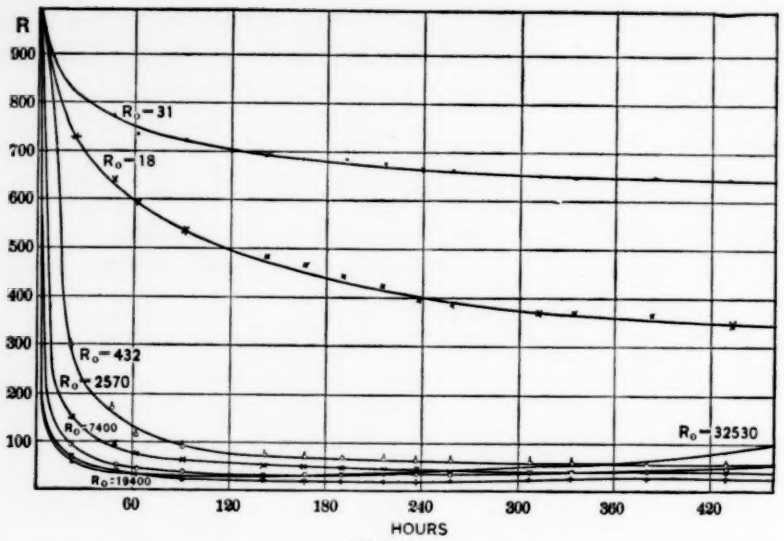

Fig. 2.

curious exception to the law and curve (3) $R_0 = 1430$ shows that the tube was not properly exhausted.

Fig. 2 shows seven curves, known in the tables as Series II, Mch. 2. For all but $R_0 = 31$ and $R_0 = 18$ the law $y = 37 + 963e^{-.69t^{.43}}$ holds. $R_0 = 31$ descends too slowly for the compari-

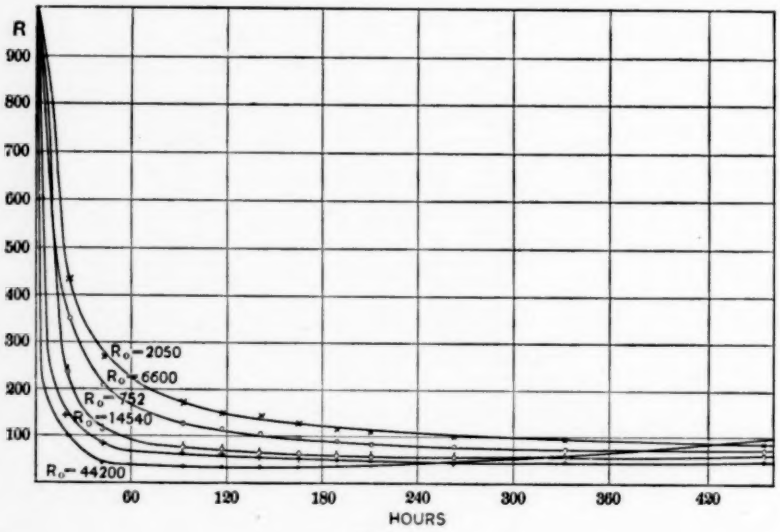

Fig. 3.

son, the higher the initial resistance the more rapid the fall. The two curves $R_0 = 32530$ and $R_0 = 19400$ rose in resistance after a few days.

Fig. 3 shows five curves, known in the tables as Series II, Mch. 4. The higher the resistance the more rapid the fall. $R_0 = 752$ is an exception. The approximate law followed is $y = 56 + 944\epsilon^{-.51t^{.43}}$. k in this case is less than k for the fresh solution, showing that the older the solution the slower the decrease.

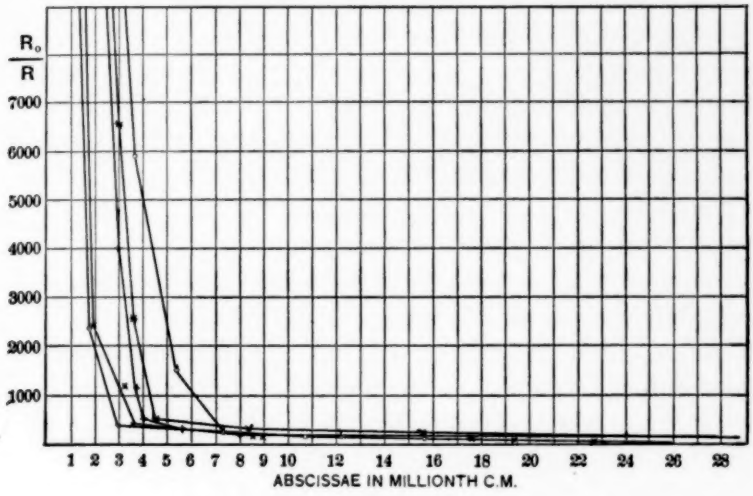

Fig. 4.

Fig. 4 shows the lack of agreement between the calculated and experimental value of the resistance for the same film. The abscissæ are thicknesses given in millionth centimeters. The ordinates are the ratios of the experimental to the calculated resistance. Just why the films kept in a vacuum decrease in resistance more slowly than those exposed to the air is not clear. In case of films between 20,000 and 50,000 initial resistance exposed to the air it was noticed that the more rapid the fall of resistance in the first forty-eight hours the more certainly could a uniform and considerable rise in resistance be expected after a few days. (*d*) and (*m*) of Mch. 22 and (*b*) of Mch. 4 are examples.

It would seem that the rearrangement of the molecules of the silver is carried on more slowly and uniformly when air is excluded,

the process being completed theoretically after an infinite time. Films kept for three months were found still falling in resistance. It certainly cannot be said that the sulphide formed on the silver due to the presence of air, retards the fall of resistance, since the resistance of films in a vacuum falls much more slowly than that of those films exposed to air. It is possible that the quicker the rearrangement takes place, the less enduring the film. There may be a greater accumulation of the silver particles in some regions than in others and a space practically free from silver particles may exist. At all events, it was noticed that films of 30,000 or more initial resistance, when heated a few minutes, showed a greater decrease of resistance, but when heated several hours increased in resistance to values higher than the initial reading. Similarly, high films as (*b*), (*d*) and (*m*) show *k* equal to .76, .85 and .80 respectively, as compared with ordinary values of from .2 to .5. This means a rapid decrease in the first twenty-four hours and is followed by an enormous increase within a week after silvering. The films which behaved in this manner were formed from fresh silver solutions.

In general, then, the decrease of resistance with time is caused by a gradual settling down of the silver molecules into a more and more compact mass. Heat accelerates this action, bringing in a few minutes the film into that condition which time acting alone accomplishes only after months. Preliminary experiments indicate that electric currents and shocks produce a similar effect upon a film as that produced by heat.

I wish to express my gratitude to Professor A. A. Michelson, who suggested this subject for research and who has forwarded its accomplishment in all ways and not least by the untiring patience and interest which he has ever shown.

THE UNIVERSITY OF CHICAGO.

(*Presented as a thesis for the degree of Doctor of Philosophy.*)

AN ELECTRIC CURVE TRACER.[1]

By Edward B. Rosa.

I.

ONE of the most interesting and fruitful methods of investigation of alternating current phenomena is the tracing of the forms and phases of alternating current curves. Transformer diagrams have revealed much of the actions and reactions which occur in the transformer; and "dynamo diagrams" and "motor diagrams" in the same way tell their story of the actions and reactions which occur between armature and field when the magnitude and character of the load varies. And when we consider the rapid multiplication of alternating current apparatus, both of single phase and multiphase varieties, it is evident that the field of usefulness of alternating current diagrams, already very great, is constantly growing.

The practicability of this method of investigation and testing has been seriously limited by the great labor of obtaining the curves, and the insufficient accuracy of the curves when obtained. To find and plot a dozen or more points and then draw a smooth curve through them and call it the curve of current or of electromotive force will sometimes answer, for want of something better. But in most cases it is unsafe to infer very much from such a curve. What is wanted is either a continuous line or so large a number of points as to be practically equivalent to a continuous line, marking exactly the fluctuations of the current. And when several such related curves have been drawn to scale on a single sheet, showing the forms, phases and relative amplitudes of the currents and electromotive forces (and perhaps also the magnetization and power waves) which are concerned in the operation of a given machine or system, we have a beautiful picture of what in algebraic language merely is not very attractive.

[1] A preliminary account of this instrument is given in the *Electrical World*, Vol. XXX., p. 631, and in *The Electrician*, Vol. XL., p. 26.

There are two general methods by which current curves have been drawn, (1) *the instantaneous method* and (2) *the step by step method*. The first named method involves the use of a light vibrating body which can be made to vibrate synchronously with the current to be reproduced, and reflect a beam of light upon a screen. Several attempts of this kind have been made with a telephone. Frölich mounted a light mirror excentrically upon the diaphragm of a telephone and caused a beam of light reflected from it to fall upon a series of revolving mirrors in such a way as to get upon a screen a stationary wave, representing the current which passed through the coil of the telephone. But the diaphragm of the telephone inserts into the wave peculiarities of its own, and so does not give a faithful representation of the current. Hotchkiss and Millis have improved very greatly upon earlier attempts. They use a very light magnetic needle, carrying a tiny mirror and having an exceedingly high rate of vibration. This needle is mounted on a short quartz fibre, and lies in a strong magnetic field. The current to be delineated passes through a coil of wire surrounding the needle, and produces a field of force at right angles to the permanent directing field. Thus, the needle makes forced vibrations of a form, amplitude and period determined by the current and the attached mirror reflects a beam of light upon a moving photographic plate. The free vibrations of the needle are relatively small, and the curve produced is a good representation of the current producing it. For such cases as the oscillating charge and discharge of a condenser, the making or dying away current in a direct or alternating circuit, this method is peculiarly adapted. But it seems doubtful whether it can be made to give several curves on a single sheet so as to indicate relative phases, and drawn sufficiently large and sufficiently accurately, to be satisfactory for general use.

Objection has been made to the step by step method because it does not give an instantaneous result; that the waves of current passing through a dynamo or a transformer are not all alike, and we do not know what we are getting when we take one point at a time. On the contrary, is it not desirable for most purposes that we should get an average result? If the waves vary slightly from instant to instant, and we take only one, we do not know how that

one varies from its fellows, nor what the average is, unless we take a very great number of waves. By the step by step method we get a fair average, and, if the curves are self-consistent, we may be sure the variations are inconsiderable. If the speed or load or electromotive force of the machine vary greatly, it is impossible to obtain satisfactory curves. But, if, by the step by step method, a curve can be obtained in a few minutes it is usually possible to keep these quantities constant during that time.

The step by step method, using an instantaneous revolving contact, has been employed by numerous observers, and the details of the process have been variously arranged. In general, the instantaneous value of the current is determined by measuring the instantaneous value of the potential difference at the terminals of a known resistance, and dividing by that resistance. Hence all current and electromotive force curves depend upon the measurement of potential. Ryan and Merritt used a special form of electrometer, with a condenser in parallel, and the steady deflection of the electrometer indicated the potential difference which existed at the instant the revolving contact maker closed the circuit. As the position of the brush was advanced step by step, the electrometer indicated the corresponding values of the electromotive force or current. Another plan is to allow the condenser to be charged by the revolving contact, and subsequently discharged through a ballistic galvanometer. The throw of the needle gives a measure of the difference of potential, which in turn is proportional to the instantaneous value of the current or electromotive force which is being studied. Still another plan is to charge and discharge the condenser by means of a rotating commutator through a galvanometer, and observe the steady deflection. Duncan's method using several electro-dynamometers and a galvanometer, and simultaneously determining as many curves as there are dynamometers, is very interesting, but in practice does not prove altogether attractive. It requires a good many observers, and I have not succeeded in getting curves of sufficient accuracy with it.

An important modification of these methods is to match the instantaneous difference of potential to be measured against a steady difference of potential due to a current flowing through a wire re-

sistance. The latter is varied until it is equal to the former, as shown by silence in a telephone connected across the circuit. The steady potential difference is then read from a voltmeter. Instead of a telephone, a galvanometer may be used. This has an important advantage over a telephone in that the direction of the deflection shows whether the adjustable resistance should be increased or decreased, and thereby greatly facilitates the operation.

I have found it advantageous to arrange the apparatus in still a different fashion. A hard rubber rod about a meter long is wound with a single layer of insulated wire, and along one element of the cylindrical surface the insulation is removed. The terminals of this

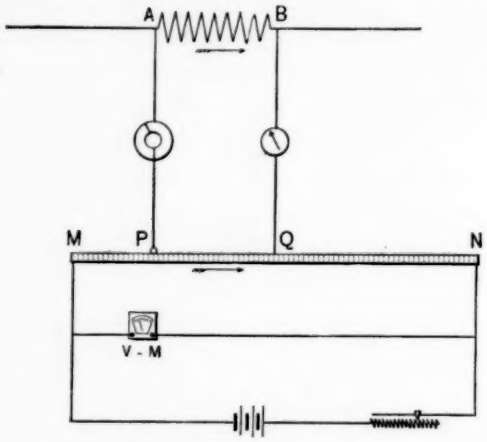

Fig. 1.

coil are joined to a battery of a few storage cells, and a current is sent through it. This spiral has so much greater length of wire than a single straight wire of the length of the rod, that it can sustain a much greater difference of potential between its ends, and so serves the purpose better than the single wire would do. Let the current to be delineated pass through the noninductive resistance AB; we have to measure the instantaneous value of the potential difference of the terminals of AB at successive instants throughout the period of a wave. B is joined through the galvanometer to the middle Q of the spiral MN. A is joined through the revolving contact-maker to a sliding contact P. When P is so adjusted that the galvanometer deflection is zero, the potential difference be-

tween P and Q is the same as that between A and B. The difference in potential of P from Q is proportional to the distance PQ; and it is positive on one side and negative on the other. If the spiral is of constant diameter and uniformly wound, and the difference of potential of its terminals is maintained constant, by means of an adjustable resistance (a voltmeter being joined to the terminals of the spiral), accurate values of the potential difference can be obtained from these scale readings PQ. Using these scale readings as ordinates and the corresponding settings of the brush on the revolving contact-maker as abscissas, a curve of current can be plotted, its scale being of course determined by the value of the constant resistance AB through which the current passes, and the value of the constant potential difference between the ends of the solenoid MN.

This method I have found much more satisfactory than condenser-electrometer methods or the usual form of the telephone method. But at the best it is slow and laborious. The successive settings of the brush on the revolving contact-maker are very tedious, if one undertakes to determine accurately a large number of points. The scale readings must all be taken and recorded, and finally the curve plotted out carefully by hand. One does this very willingly for a while, but if one must take a good many curves, each of a hundred or more points, the work becomes a burden.

It was while engaged in this kind of work, more than a year and a-half ago, that I undertook to devise an apparatus that should reduce the labor and increase the speed of working. This could clearly be accomplished by doing away with the three most laborious parts of the work, viz: (1) the setting of the brush, (2) taking and recording the scale readings, and (3) plotting of the curve. Until these operations could be performed automatically, the process of curve tracing would continue to be slow and laborious.

When one's attention is thus explicitly directed to the matter, a method of proceeding is at once suggested. Take a pantagraph (Figure 2) and fix one end at P' and join the other to the contact piece P, which travels up and down the solenoid MN. Let the bar EE' of the pantagraph carry a pencil at F. As P is moved to and fro upon the solenoid, the point F travels to and fro along a line parallel to AB at a reduced speed. The excursions of F bear the

same ratio to those of P that the distance $P'F$ bears to $P'P$. Between the successive settings of P, which are made every time the brush of the contact-maker is advanced, let the plate R^1R^2 be advanced a step toward MN, but not until the pencil has been pushed down upon the paper covering R^1R^2 and a dot made. In this way we shall obtain a current curve plotted out upon a plain sheet.

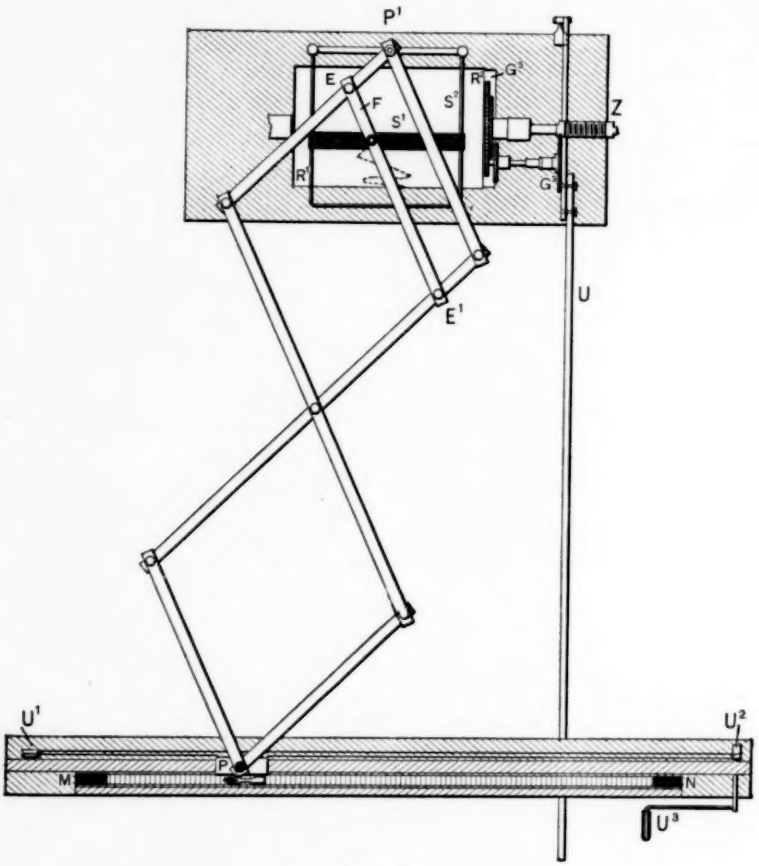

Fig. 2.

In order, however, to facilitate the operation let the paper be wrapped upon a cylinder, and the latter advanced by a ratchet wheel and gearing, one tooth at a time. Also, let the pencil be operated by an electromagnet. Further, let the brush-holder and brush of the contact-maker be advanced simultaneously with the cylinder by

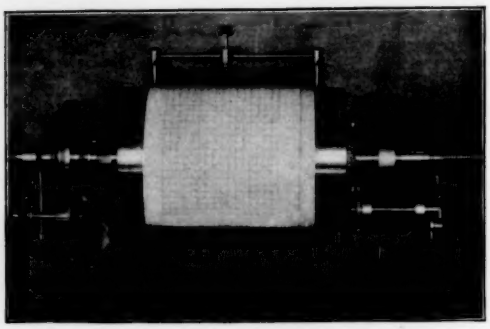

Fig. 3. Recording Cylinder and Contact-Maker (First Instrument).

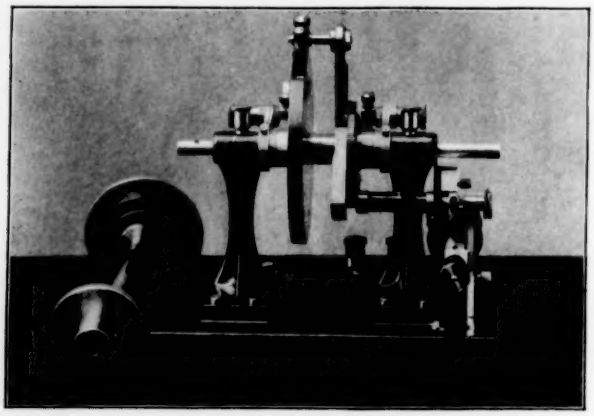

Fig. 7. The Contact-Maker.

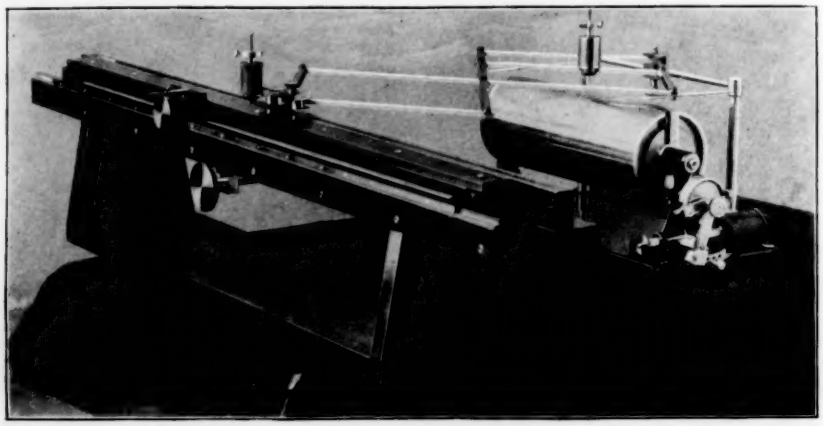

Fig. 8. The Measuring and Recording Apparatus.

gearing them together. Then arrange so that lifting a lever will close the circuit of the magnet at *F* and print the point; dropping the lever will advance the cylinder and brush ready for the next setting. Then operating this lever with one hand and the movable contact *P* with the other, we shall be prepared to draw current curves with neatness and despatch.

Such considerations as these led to the construction of the instrument shown in figure 3. It was not, however, designed exclusively for curve tracing purposes, but was to be a "universal chronograph," capable of turning at widely varying speeds, and especially useful for tuning fork experiments. A horizontal screw in front of the cylinder was driven by gears from the shaft of the cylinder, and this screw propelled a carriage which was to carry a tuning fork or a recording pen, or both. The cylinder was also to rotate continuously when used for curve-tracing purposes, the original plan being to have the brush advance slowly and continuously, and by keeping the contact *P* moving so as to have the spot of light on the galvanometer scale always on the zero line, the current wave would be represented by a continuous line traced out upon the cylinder. But the spot of light could not be kept on the zero unless the cylinder turned with exceeding slowness, and hence the plan was modified. The step by step method was adopted, the original driving gear was abandoned and a ratchet and pawl substituted, the screw and carriage of the "universal chronograph" were for the time removed, and the instrument became an electric curve tracer. This explanation will account for some peculiarities of the instrument which otherwise might be thought to be evidence of bad design.

II.

Description of the Instrument.

The brass cylinder *RS* (Fig. 5) is 8 inches (20 cm.) in diameter and 12 inches (31 cm.) in length. It is mounted upon bushings or sleeves *JJ'* which turn in the bearings *JJ'*; in these sleeves runs a five-eighth-inch steel shaft. This shaft is in line with the shaft of the dynamo, from which current is to be taken, and is joined to it by a half-inch rod and two flexible couplings, of which one (*K*) is

shown. The connecting rod carries at its end a thin, steel disc or diaphragm, as shown, and this is screwed to a ring which is fastened to the shaft of the instrument by a set screw. Thus the couplings allow slight end play and obviate the necessity of perfect alignment of the instrument with the dynamo, but the connection is perfectly rigid so far as rotation is concerned.

The gear G' is rigidly connected with the sleeve or hub of the cylinder. Next to it is a gear G^2 of half its diameter and number of teeth which is joined to the disc H; G^2 and H turn loosely upon the sleeve J. By means of the counter gears shown in figure 3 the disc H, which carries the brush-holder Q, may be caused to rotate four times as fast as the cylinder. But by throwing out the counter gears and inserting a pin through G', the latter is clamped to G^2 and H, and the brush-holder and cylinder advanced together at the same angular speed. L is a fibre disc mounted upon an inner shaft, and hence driven with the speed of the armature of the dynamo. Into its circumference is set a steel knife edge, which is connected electrically with an insulated slip ring T', upon which a contact brush (not shown in the drawing) rests. Another contact brush rests upon a brass circle T^2 which is screwed to the hard rubber ring H^2; to the circle T^2 the brush-holder is attached. These two brushes are on the back of the instrument and do not appear in figure 3. When the dynamo is running and the disc L is in rapid rotation, once in each revolution the knife edge comes under and touches the brush carried by the brush-holder Q. The current which enters at the slip ring T' flows up through the wire to the knife edge and brush, thence to the ring T^2 and away to the galvanometer.

At the other end of the cylinder is the lever U (figure 2) which extends forward to the measuring solenoid MN and its end is held in the operator's left hand. This lever carries a pawl which engages the teeth of the ratchet wheel G^3. The pinion G^4 is carried on the same shaft with the ratchet and engages the large gear G^5 which is rigidly joined to the cylinder. There are 60 teeth in the ratchet, and twelve times as many in G^5 as in G^4; hence one tooth of the ratchet advances the cylinder $\frac{1}{720}$ of a revolution, or half a degree.

If the dynamo is a two-pole machine (as is the case with one of

the dynamos with which the curve-tracer has been used) we use the counter gear and advance the brush four times as rapidly as the cylinder. One tooth then gives a step of 2° in the wave, and hence there are 180 steps in one complete revolution of the brush and 180 points plotted in one complete wave, which extends through one-quarter of the circumference of the cylinder. By changing the stop under the lever so that two teeth are taken at each step we get 90 points in a wave-length, three teeth at a time give 60 points in a wave-length, etc. We usually take one or two teeth at a step.

If the dynamo is an eight-pole machine (as is the case with our small alternator), we throw out the counter gears and advance the brush and cylinder at the same angular rate. A quarter of a revolution gives one complete wave, so that the curve is drawn on the same horizontal scale, as before, and with 180, 90, or 60 points in each wave-length according as each step is one, two or three teeth of the ratchet. By changing the counter gears, dynamos of different numbers of poles can be used and all curves drawn on the same horizontal scale.

A cord attached to the sliding contact piece P of the measuring solenoid passes over a pulley at the left hand end of the solenoid and is wound over a small drum at the right hand end, a crank being attached to this drum. By turning the crank one way or the other the sliding contact is moved to and fro and exact settings quickly and easily made. The galvanometer is a very quick, dead-beat D'Arsonval, and the scale is directly over the solenoid, attached to V, figure 4. The crank handle is turned with the right hand until the spot of light is on the zero; no current is then passing through the galvanometer and the balance of the potentials of AB and PQ is exact. The lever is then lifted, and this closes a key (figure 4) which sends a current through the small electro-magnet on the pantagraph. This throws a steel point down upon a type-writer ribbon which is held on a frame over the cylinder, and prints a point on the paper beneath. The lever is then dropped and the pawl N^2 having engaged a tooth of the ratchet wheel, the cylinder and the brush of the contact-maker are advanced a step. The spot of light on the scale thereupon jumps off to the right or left from the zero mark; the crank is turned and the contact P moved in the

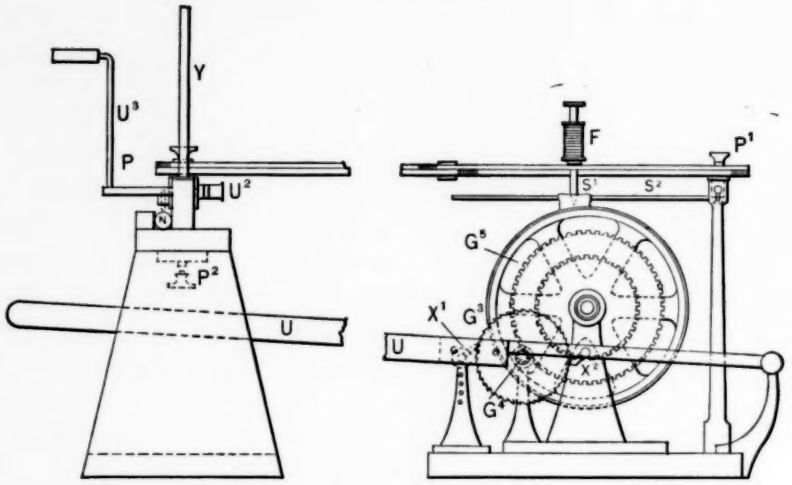

Fig 4.

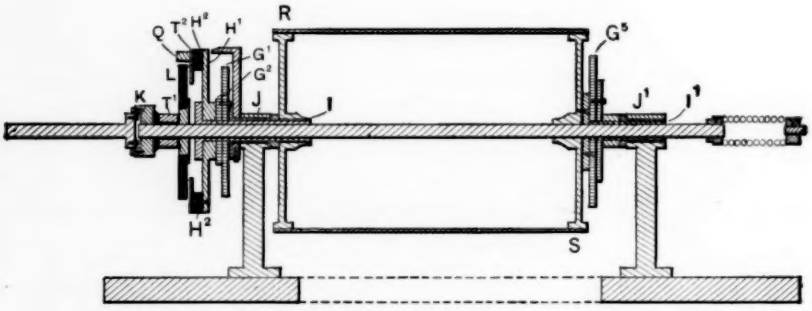

Fig. 5.

direction the spot has gone until the latter is brought back to zero. The lever is again lifted and a point printed; then dropped, and the cylinder and brush advanced as before. The operator meanwhile keeps his eye only on the spot of light on the scale, and does not see the points printed under the ribbon, or the sliding contact under the solenoid. Since the extent of the deflection of the needle indicates about how far the handle must be turned to bring it back to zero, the settings may be made very rapidly, using a quick dead-beat galvanometer. Indeed it is no uncommon performance to print off several hundred points at the rate of *twenty points per minute*, and yet take time enough to make accurate settings and obtain a beautiful curve.

If a dynamo would run at a constant speed and yield a constant electromotive force one operator could do the whole work. But usually one assistant keeps the speed and voltage steady by hand regulation.

One of the greatest difficulties of curve tracing by any method when an instantaneous contact is used is in keeping the contact in good order. A short and yet certain contact, of nearly uniform duration, is difficult to secure. In this instrument a spiral spring presses the toe of the brush down upon the disc. This insures good contact and prevents vibrations of the brush as it passes over the knife edge, which should project above the fiber very slightly.

The galvanometer should have a very constant zero, should be dead-beat and of very short period. To secure these points and yet have it sufficiently sensitive would be difficult if a single wire were used as on a slide wire bridge, seeing the galvanometer circuit is closed only about $\frac{1}{500}$ part of the time. But with the solenoid, which consists of about 925 turns of copper wire in 100 centimeters, having a total length of over 20 meters, a fall of potential of 5 volts is maintained. Settings are usually made to within a millimeter on the solenoid, and any error of setting is by the pantagraph reduced in the paper to one-eighth its magnitude. The fall of potential through the solenoid is maintained constant by a voltmeter and rheostat, and any heating of the wire causes no error. If there is no appreciable leakage by imperfect insulations, if the spiral is uniformly wound and if the resistances are non-inductive and their true values known the accuracy of the curves is assured.

When the curves for a transformer are to be drawn the connections are to be made as in figure 6. The character and magnitude of the load may be altered without changing the connections, but the values of the resistance may require changing so as to bring the curve to a suitable scale. Having first drawn the primary current curve with the wires CD of the switch joined to the points AB (the resistance AB being removed) the switch is thrown over to connect PQ to the non-inductive resistance R^2, the difference of potential A^2B^2 being proportional to the whole difference of potential of the primary circuit. This gives the primary electromotive force curve. To draw the secondary electromotive force the resistance R^2 is joined

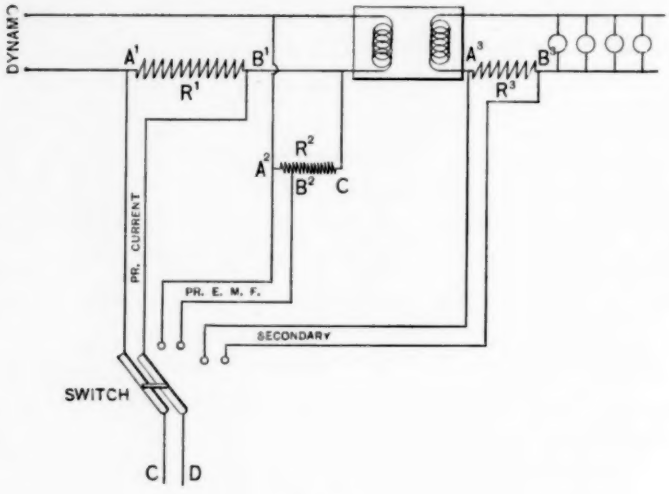

Fig. 6.

to the secondary of the transformer. To draw the secondary current the switch is thrown over to the third position.

If MN is 40 inches and the difference of potential is 5 volts, then 1 volt corresponds to eight inches. The pantagraph reduces the scale in the ratio of eight to one, hence one inch on the paper corresponds to one volt on the solenoid. If the resistance R' is $\frac{1}{10}$ ohm, then 10 amperes will cause a difference of potential of one volt at its terminals, and this will be represented on the paper by an ordinate one inch high. Hence the scale of this current curve would be 10 amperes per inch. If the resistance is $\frac{1}{5}$ ohm, then the scale is 5 amperes per inch, etc. If R^2 is 2,000 ohms and A^2B^2 is 20 ohms, that is $\frac{1}{100}$ of the total, then since the difference of potential of A^2B^2 is drawn out on the paper on the scale of one volt per inch, the scale of the electromotive force curve will be 100 volts per inch. If A^2B^2 is 10 ohms, then the scale is 200 volts per inch,[1] etc. In this way several curves may be drawn on the same sheet, each to a suitable predetermined scale, and their relative positions on the paper give their relative phases.

[1] I have used inches instead of centimeters in this work because the most suitable cross-section paper I could obtain is ruled in twentieths of an inch.

III.

An Improved Form of the Curve Tracer.

The chief objection to the instrument as above described is that it must be placed near to the generator of the electric current which is to be delineated, unless a synchronous motor be used; and in that case a special form of motor would probably be required, in order to secure, what would be absolutely essential, a very close synchronism between generator and motor. In order to avoid the necessity of placing the measuring and recording apparatus nearer to the generator than may be convenient, and also to avoid moving it as different dynamos shall be used in succession, a new form of the apparatus was devised in which the contact-maker was separated from the remainder of the apparatus. (See Figures 7 and 8.) Instead of advancing the brush of the contact-maker by joining it mechanically with the recording cylinder, the brush is advanced step by step by a ratchet and pawl and intermediate gearing. The pawl is actuated by an electro-magnet, the current in which is made and broken by the operator at the measuring and recording apparatus which may be some distance away. The same current which advances the brush also advances the recording cylinder, by a mechanism similar to that on the contact-maker. Thus the brush and cylinder move simultaneously, the length of the step of each being limited by the positions of the stop pins; and these positions are determined by the number of points one wishes to print in a wave-length, and the number of poles there may be in the generator which is employed, as in the first form of the apparatus.

The Contact-Maker.

The shaft of the contact-maker is joined to the shaft of the dynamo by a coupling shown in the figures. At the nearer end of the connecting rod the latter is joined to the plate (which is fastened rigidly to the shaft by a collar and set screw as shown) by a flexible steel diaphragm, so that perfect alignment of the contact-maker and dynamo is not necessary. At the farther end, the large circular plate carries two steel pins, one at its centre and one near its edge, which fit loosely into two corresponding holes in a plate clamped to the end of the dynamo shaft. Thus the end-play of the armature is

not communicated to the contact maker; and if the latter is clamped to a firm base, the shaking and jarring of the dynamo are not transmitted to the contact-maker. Two oil cups carry a good supply of oil, and the contact-maker runs as smoothly as a top, hour after hour, with very little attention.

The galvanometer current, which flows momentarily when the knife edge in the hard rubber disc passes under the contact brush, enters at the binding post which is seen behind the left hand oil cup, passes through a brush to a slip ring on the left of the large disc and from there to the steel knife edge or contact piece. This is held in place by three screws, and can be adjusted so as to give the most perfect contact. It should project slightly above the surface of the rubber. Passing from the brush and the insulated brush-holder, the current flows from a second slip ring to a second brush and off from the second binding post on the right. The mechanism which advances the brush-holder can be understood by comparing figures 7 and 8. The teeth of the ratchet and large gear are numbered so that the initial position of the brush can be recorded, and the brush returned to the same position for the beginning of successive curves. The milled head on the end of the spindle which carries the ratchet wheel and pinions enables one to set the brush quickly by hand at any desired place. The two binding posts seen at the back of the base are the terminals of the coil of the electro-magnet which operates the rachet.

The Measuring and Recording Apparatus.

This is shown in figure 8, the two principal parts being connected together by the pantagraph, which is made of brass. Over the cylinder, and carried by one of the bars of the pantagraph, is the printing electro-magnet. The type-writer ribbon (not shown) is held in place under the magnet by two rods, which extend forward from the uprights behind the cylinders. The free end of the pantagraph is attached to a carriage which is moved to and fro by means of the lower of the two milled heads at the front of the wooden frame; a cord tied to the carriage passes over a pulley at either end and is wound around a drum below. The distance of the printing point from the fixed end of the pantagraph is, in this instrument, one-fifth

of the length of the pantagraph, and hence the excursions of the printing point are one-fifth of those of the carriage. A second printing point mounted on the carriage prints a point on a strip of paper (if it is desired) whenever the first prints a point on the sheet of paper wrapped upon the cylinder. One can then duplicate a curve without running the dynamo, the settings of the movable contact carried by the carriage being made by the dots printed on the strip of paper. Curves may, of course, be duplicated at any time, and curves originally drawn on separate sheets may be combined on the same sheet.

The contact key and connecting wires are not shown in the figure, but connections are made in a manner similar to that indicated in figures 1 and 6.

IV.

The Curves.

Figure 9 shows four curves drawn in August, 1896, using a small, eight-pole Westinghouse alternator. The generator was joined to the primary of a transformer, and to the secondary of the transformer was first joined a coil of wire having a fine iron wire core, and then the coil was replaced by a Stanley condenser having a capacity of about 7 microfarads.

Curve I is the electromotive force of the secondary of the transformer, impressed upon the inductive coil, and it is precisely similar in form to the electromotive force of the generator. The peaks or ripples which appear in this curve are due to the teeth in the armature of the dynamo. The induction through the air gap from the pole pieces to the armature is not uniformly distributed, but is tufted, and the induced electromotive force shows fluctuations of a corresponding periodicity. The distance apart of the teeth in this case is one-seventh of the distance between two consecutive poles ; and hence the distance apart of these peaks is one-seventh of half a wave-length, or one-fourteenth of a whole wave-length. But since the positive and negative half waves are alike in form as well as in average value, there can be no even harmonics present in the wave. These ripples in fact consist chiefly of two odd harmonics, one having thirteen and the other fifteen times the frequency of the funda-

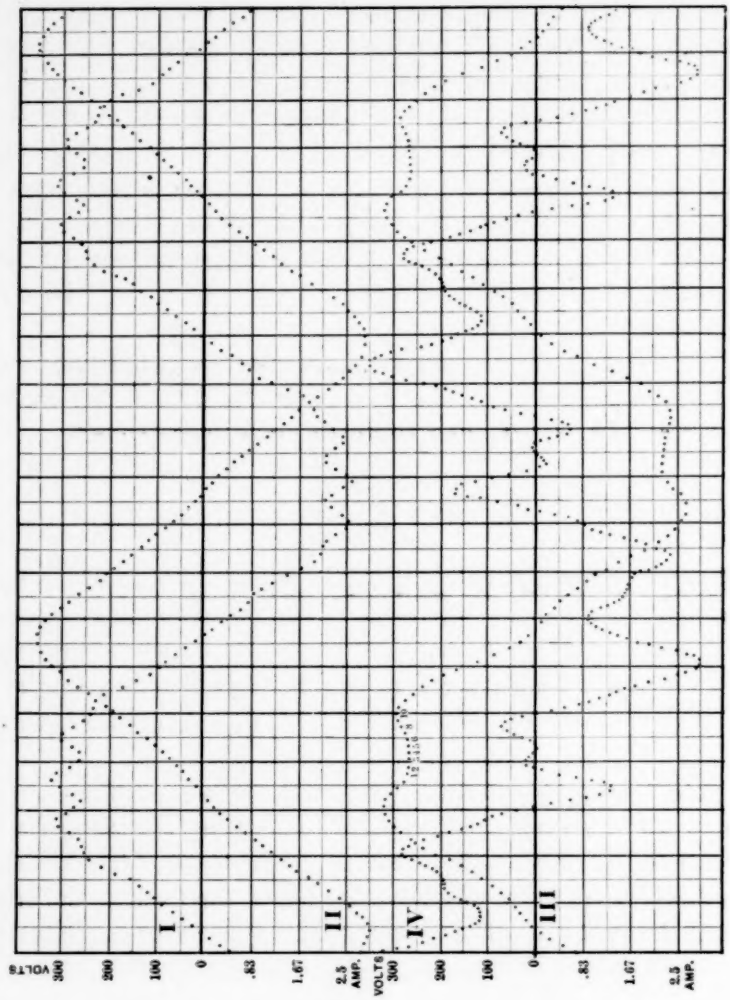

Fig. 9.

mental. But apart from these higher harmonics, which are attributable to the teeth of the armature, the curve would not be a simple sine curve, and upon analysis we find three other harmonics of frequencies 3, 5 and 9 times the fundamental. The equation of the curve is:

$$\begin{aligned} e = {} & 30.66 \sin(x - 0^\circ\ 56') + .80 \sin(3x - 6^\circ\ 17') \\ & - 1.65 \sin(5x - 14^\circ\ 22') + .40 \sin(9x + 7^\circ\ 50') \\ & + 1.11 \sin(13x - 30^\circ\ 45') - 1.18 \sin(15x - 38^\circ\ 40'). \end{aligned} \qquad (1)$$

Curve II is the current flowing through the inductive coil. The effect of induction is to damp out of the current to a greater or less extent the harmonics of the electromotive force; and to damp out the highest harmonics most completely. On the other hand, the effect of an iron core, because of its variable permeability during a cycle of magnetization, is to put into the current some upper harmonics, especially that of frequency three times the fundamental. Hence, curve II shows the third harmonic[1] strongly, while all others are very small.

The equation of the curve is:

$$\begin{aligned} i = {} & 30.49 \sin (x - 1^\circ 8') - 4.14 \sin (3x - 23^\circ 13') \\ & + .38 \sin (5x - 51^\circ 21') - .15 \sin (9x - 34^\circ 45') \\ & + .12 \sin (13x - 48^\circ 23') - .14 \sin (15x - 10^\circ 12') \end{aligned} \quad (2)$$

The coil being now replaced by the condenser, the current and electromotive force are both found to have changed their form. The dynamo is otherwise unloaded, and the armature reaction of this current is considerable. The electromotive force (curve III) has nearly lost its highest harmonics, while the third and fifth are greater than before. The equation of the curve is:

$$\begin{aligned} e = {} & 31.43 \sin (x - 3^\circ 40') + 1.38 \sin (3x - 18^\circ) \\ & - 3.46 \sin (5x - 55^\circ) - .38 \sin (9x + 2^\circ) \\ & - .25 \sin (13x + 20^\circ) + .18 \sin (15x). \end{aligned} \quad (3)$$

Curve IV is the current flowing into the condenser. As can be seen at a glance, its fifth harmonic is very pronounced and gives the curve a very striking form. The 3d, 5th, 13th and 15th harmonics are also present, and in nearly the same proportion, while the 7th is represented very slightly. Of course, the 7th harmonic could not be present in the current if it were not present in the electromotive force, but its magnitude in the latter is too minute to detect with certainty. The equation of curve IV is:

$$\begin{aligned} i = {} & 21.05 \sin (x - 20^\circ 16') + 2.66 \sin (3x - 73^\circ 49') \\ & - 12.40 \sin (5x + 36^\circ 37') \\ & + .36 \sin (7x - 82^\circ) - 2.49 \sin (9x + 30^\circ 22') \\ & - 2.08 \sin (13x - 48^\circ 30') - 2.08 \sin (15x - 87^\circ 32') \end{aligned} \quad (4)$$

[1] For brevity I shall speak of the harmonic having three or five times the frequency of the fundamental as the third or fifth harmonic, etc.

The ordinates of curve IV are everywhere proportional to the rate of change of curve III. In other words, curve IV is the derived curve of III. This is readily seen from the fact that the current flowing into or out of a condenser is proportional to the rate of change of the difference of potential of its terminals; that is: $i = C\frac{de}{dt}$. By inspecting the curves it will be seen that wherever curve III is increasing, the ordinates of curve IV are positive; and wherever the electromotive force is decreasing the current is negative. Also curve IV crosses the zero line wherever curve III has a maximum or minimum. To show how beautifully these two curves correspond, several points on the top of curve III have been numbered. At point 1 the electromotive force is still diminishing, but not so rapidly as before, and the current is decreasing. At 2 the electromotive force has ceased to decrease and the current is zero. At 3 the electromotive force has slightly increased, and the current is small but positive. At 4 the E. M. F. is still increasing, but more slowly, and the current is diminishing. At 5 the E. M. F. curve is level, and the current has fallen to zero. At 6 the E. M. F. is increasing again and the current is again positive. At 8 the *slope* of the E. M. F. curve is a maximum and the current is a maximum. At 10 the E. M. F. has reached a maximum and the current has fallen to zero.

Suppose we consider that each one of the harmonic components of the electromotive force acts independently of all the others, producing its own current. Then knowing the capacity of the condenser and the period of the current, we could calculate all the harmonic components of the current, knowing the components of the electromotive force, or *vice versa*. Let us proceed to do the latter:

Take e_1, e_3, e_5, etc., as the instantaneous values of the components of the electromotive forces.

E_1, E_3, E_5, etc., as the maximum values of the same.

i_1, i_3, i_5, etc., I_1, I_3, I_5, etc., the corresponding values of the current.

C=capacity, $p=2\pi n$, φ=phase angle.

$$\text{Then}\quad e_1=E_1\sin(pt+\phi_1),\quad i_1=C\frac{de_1}{dt}=pCE_1\cos(pt+\phi_1),\quad I_1=pCE_1$$

$$e_3=E_3\sin(3pt+\phi_3),\quad i_3=C\frac{de_3}{dt}=3pCE_3\cos(3pt+\phi_3),\quad I_3=3pCE_3$$

$$e_5=E_5\sin(5pt+\phi_5),\quad i_5=C\frac{de_5}{dt}=5pCE_5\cos(5pt+\phi_5),\quad I_5=5pCE_5$$

$$\vdots\qquad\qquad\vdots\qquad\qquad\vdots$$

$$e_{15}=E_{15}\sin(15pt+\phi_{15}),\quad i_{15}=C\frac{de_{15}}{dt}=15pCE_{15}\cos(15pt+\phi_{15}),\quad I_{15}=15pCE_{15}$$

$$i=i_1+i_3+i_5+\ldots\ldots\ldots\ldots+i_{15}$$

$$\text{or } i=pCE_1\cos(pt+\varphi_1)+3pCE_3\cos(3pt+\phi_3)+5pCE_5\cos(5pt+\phi_5)\ldots\ldots\ldots\ldots\ldots$$
$$+15pCE_{15}\cos(15pt+\varphi_{15})$$

Thus we see why the higher harmonic components of the current are so much greater than those of the electromotive force, the 15th being 15 times as great in proportion to its electromotive force as the fundamental.

From the above equations we have:

$$\frac{E_1}{I}=\frac{3E_3}{I_3}=\frac{5E_5}{I_5}=\ldots\frac{15E_{15}}{I_{15}}=\frac{1}{pC}$$

from the first terms of the equations (3) and (4) above we see that $\frac{1}{pC}=\frac{31.43}{21.05}=1.50$ very nearly. Using this value of $\frac{1}{pC}$ and the values of I_3, I_5, etc., from equation (4) we will calculate the Es.

$$E_n=\frac{\frac{1}{pC}}{n}I_n=\frac{1.50}{n}\cdot I_n$$

N	$\frac{1.50}{n}$	I_n	E_n (calculated)	E_n found by analysis.
3	.50	2.66	1.33	1.38
5	.30	12.40	3.72	3.46
7	.21	.36	.07	
9	.17	2.49	.42	.38
13	.11	2.08	.24	.25
15	.10	2.08	.21	.18

The agreement between the values of the coefficients of the equation of the electromotive force as calculated from the current and as found by analyzing the curve of electromotive force is very close; the

difference being appreciable only in the fifth harmonic, and there it is less than one per cent. of the fundamental. The last three terms have coefficients which average *less than a hundredth part of the fundamental*, and yet the curve tracer has picked them up and placed them so exactly where they belong, that we get nearly the same values by analyzing either curve. This is a most severe test of the work of the curve tracer.

The accuracy and beauty of curve IV will be further appreciated by comparing the first part with the last part, one wave-length distant.

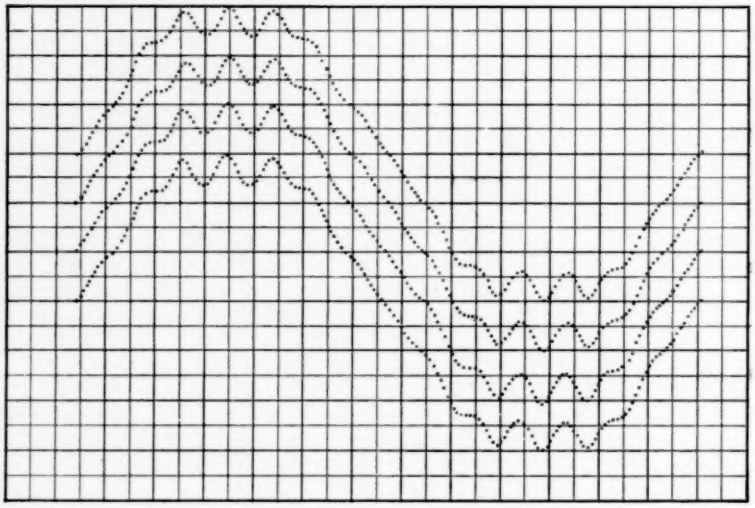

Fig. 10.

Figure 10 shows the four successive waves generated by one revolution of the eighth-pole alternator. The brush of the contact-maker is advanced exactly 90° for each wave. These four waves which, for convenience of comparison and of printing, are placed under one another instead of in succession, are so nearly alike that one can scarcely detect any difference between them. These curves not only show how symmetrical the generator is, but also show how perfectly the instrument follows the fluctuations in the electromotive force. One can readily follow the ripples down the side of the waves, although their amplitude here becomes very small. The

two harmonics of frequencies thirteen and fifteen times the fundamental reinforce each other at the summits of the fundamental and produce ripples of double amplitude. On the side of the waves, however, midway between the summits, the two waves are in opposite phases, and nearly annul each other.

In figure 11, the first curve is a repetition of one of figure 10.

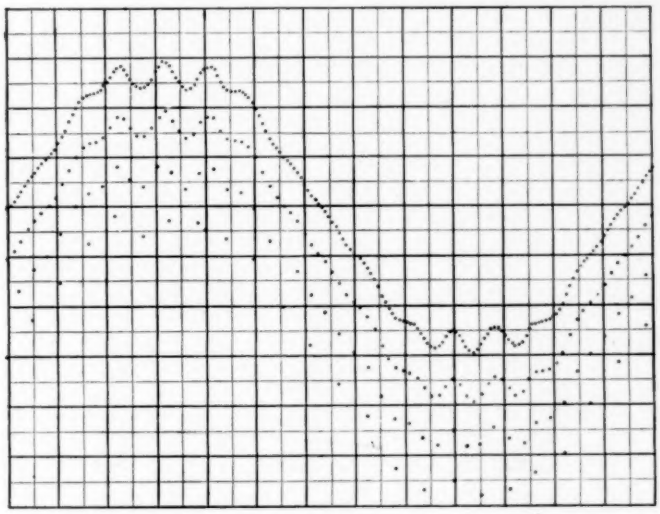

Fig. 11.

The second is the same curve with every other dot omitted. The third has only half as many dots as the second, and the fourth half as many as the third. The 23 dots of the fourth curve are just as accurately placed as those of the first curve, being every eighth dot of the first curve. Through these 23 accurately located points a curve might be drawn, but it would not very closely resemble the first curve. This illustrates the necessity of a large number of points in delineating any but very simple curves. And the want of symmetry sometimes observed in current curves is probably due to imperfect tracing quite as much as to an imperfect generator.

Figure 12 shows four electromotive force waves which very closely resemble each other. There are three transformers in series as shown in figure 13.

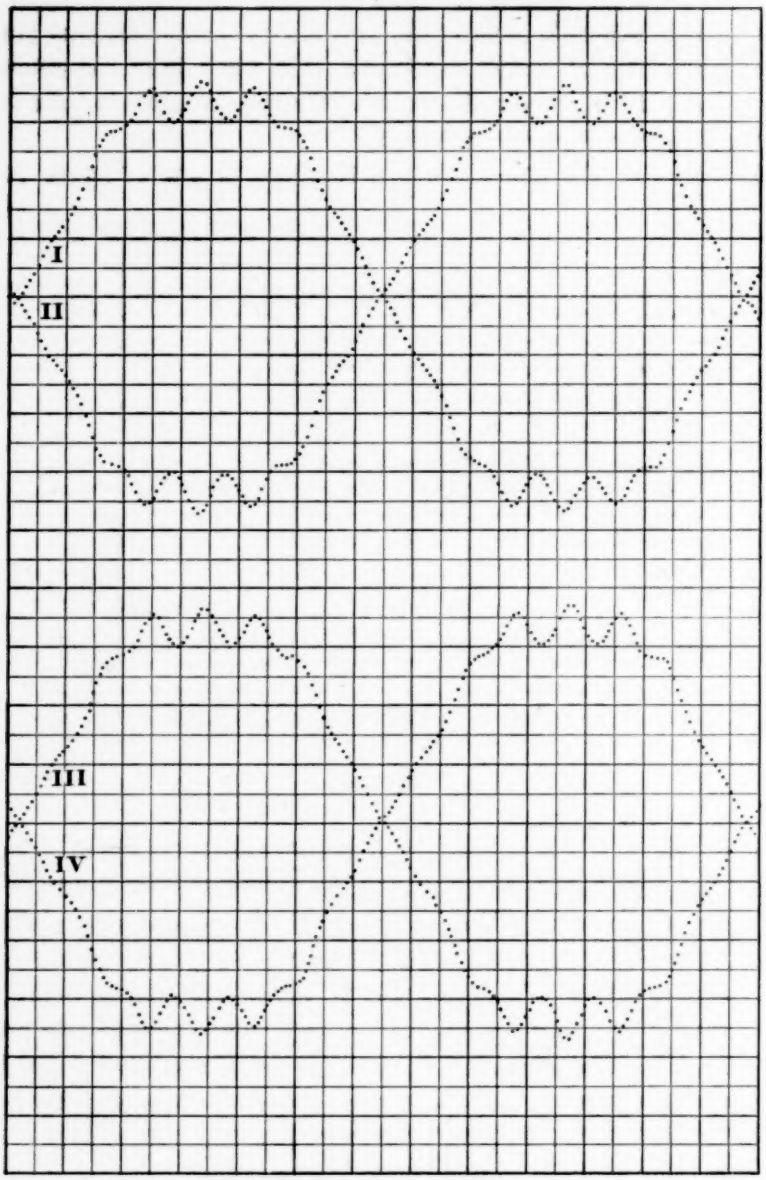

Fig. 12.

Curve I is taken by connecting the curve tracer to *ab*, and is the electromotive force impressed upon the first transformer. Curve

II is taken at *cd*, III at *ef*, IV at *gh*. Thus, as the wave of electromotive force is propagated through the chain of magnetic circuits it

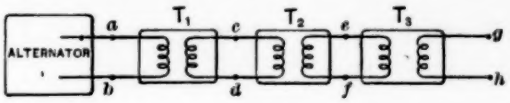

Fig. 13.

is exactly reproduced, and the phase of each wave is almost exactly opposite to the one before or behind it. There is a slight diminution in magnitude, due to magnetic leakage and resistance.

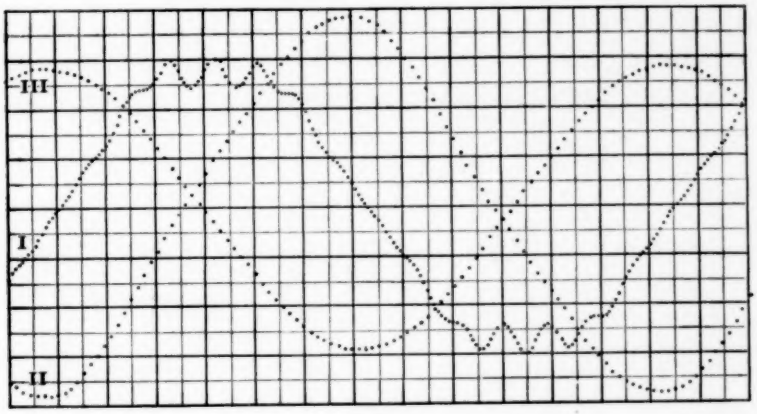

Fig. 14.

Figure 14 shows the primary (II) and secondary (III) currents of a transformer when the load is inductive. The former lags about 72° behind the primary electromotive force (I), while the secondary lays 84° behind the position opposite the primary electromotive force which it would take if there were no inductance. The coil has no iron core, and the curves are very nearly simple sine curves, the harmonics being almost completely wiped out. To see the precise reason for this we have only to write down the equation for the current in terms of the electromotive force and inductance.

$$i = \frac{E_1}{\sqrt{R^2 + p^2L^2}} \sin(pt + \varphi_1) + \frac{E_3}{\sqrt{R^2 + 9p^2L^2}} \sin(3pt + \varphi_3) +$$

$$\frac{E_5}{\sqrt{R^2 + 25p^2L^2}} \sin(5pt + \varphi_5) \ldots + \frac{E_{15}}{\sqrt{R^2 + 225p^2L^2}} \sin(15pt + \varphi_{15}).$$

But R is very small in comparison with pL, and hence we may neglect it and write

$$i = \frac{E_1}{pL}\sin(pt+\varphi_1) + \frac{E_3}{3pL}\sin(3pt+\varphi_3) + \frac{E_5}{5pL}\sin(5pt+\varphi_5)\cdots + \frac{E_{15}}{15pL}\sin(15pt+\varphi_{15}).$$

Thus we see that the 15th harmonic in the current is 15 times less in proportion to the value of the corresponding term in the electromotive force than the fundamental. In other words, the impedence for the 15th harmonic is 15 times as great as for the fundamental, and similarly for the other harmonics. This is just the opposite case to the condenser current, where the impedence is 15 times less for the 15th harmonic, and the current is correspondingly augmented.

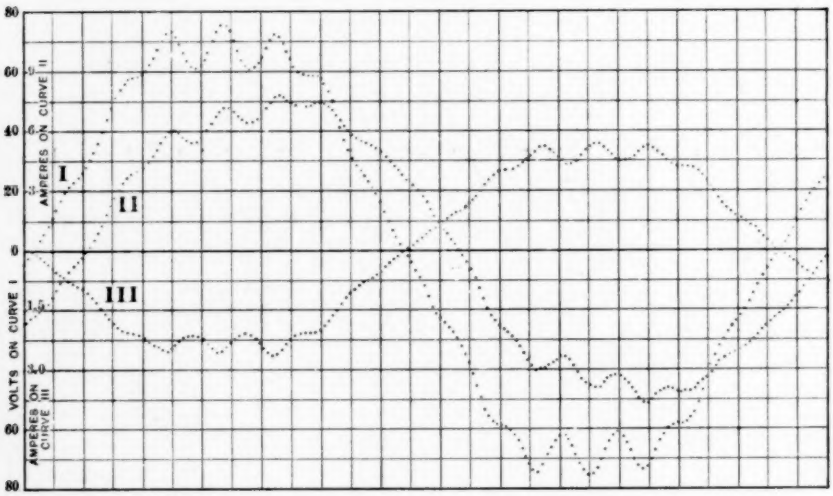

Fig. 15.

An intermediate case between these two extremes appears in figure 15, where the load of incandescent lamps is non-inductive and the resistance is the same for each harmonic. Consequently, the various upper harmonics are found in the current in the same proportion as in the electromotive force. Curve II is the primary current and III is the secondary. There are only four 16 c. p.

lamps on the secondary and hence the magnetizing current in the primary coil appreciably affects the form of the primary curve.

Figure 16 shows the curve of electromotive force of the dynamo

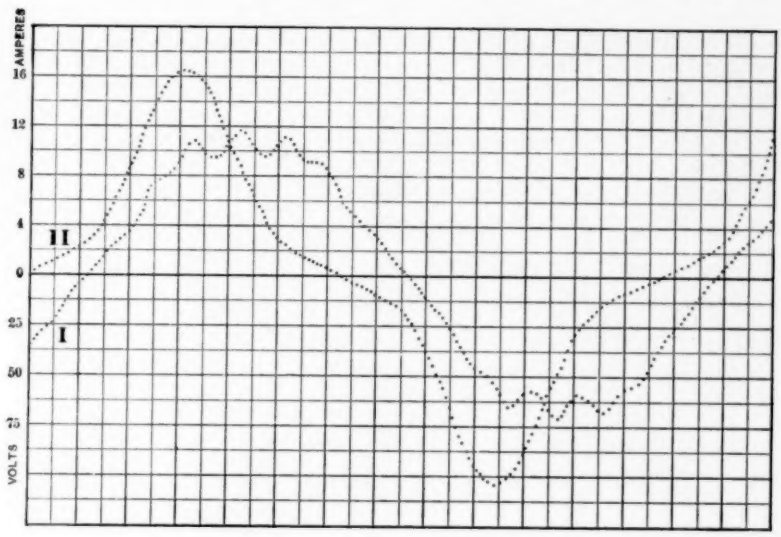

Fig. 16.

and the current flowing under this impressed electromotive force when a coil having an iron core is joined in series with a condenser having a capacity of about 20 micro-farads. The voltage of the dynamo is about 50, but by resonance between the coil and condenser it is raised to fully 400 volts both on the coil and on the condenser, and the induction through the iron coil is carried up to about 24,000 lines per sq. cm. The maximum value of the current is 16.5 amperes and it is about 17° ahead in phase of the electromotive force, instead of being nearly 90° behind, as in the case of curve II of figure 9 where no condenser was used. The effect of the third harmonic in elevating the maximum part of the curve is here very pronounced, the reason, of course, being that the magnetization is carried so far beyond the knee of the magnetization curve that the permeability falls to a low value.

The curves of figures 15 and 16 are not so perfect as the others, having been drawn under less favorable circumstances. But they illustrate the points in question satisfactorily and incidentally show what a relatively poor curve is, drawn on this instrument.

In conclusion I wish to express my obligation to Mr. O. S. Blakeslee, the accomplished mechanician of the college, for his assistance in designing the mechanical features of the curve tracer, and for his skill in constructing the instrument. And also to Mr. R. H. Hopkins, M.A., for his assistance while a graduate student and assistant in my laboratory.

WESLEYAN UNIVERSITY, MIDDLETOWN, CONN.

August 1, 1897.

ON THE THEORY OF MAGNETO-OPTIC PHENOMENA[1]. I.

By C. H. Wind.

Introduction.

MANY theories have hitherto been advanced in explanation of magneto-optic phenomena. The only one, however, completely answering the purpose, viz: Goldhammer's,[2] is unfortunately not founded in the simplest way possible on the principles of the electromagnetic theory of light, and has as one of its foundations a hypothesis[3] for which a physical interpretation is not easy to give. Another theory *not* perfectly in agreement with experience is the one of Lorentz.[4] Two or three years ago the present author showed[5] how the fundamental hypothesis in Lorentz's theory might be modified so as to bring this theory into agreement with the results of observation. This modification is not entirely free from a rather arbitrary element, and the whole theory, being many years older than Goldhammer's, starts from Helmholtz's equations of motion for electricity, which perhaps are growing somewhat obsolete at this time. For these two reasons we cannot feel entirely satisfied with the Lorentz theory.

The object of the present paper is in the first instance to show how we may start from Maxwell's equations in their very simplest form and then have only to suppose a peculiar relation between electric current and electric force in a magnetic field, in order to obtain a theory which offers a complete mathematical description of

[1] Extract from: "Eene studie over de Theorie der magneto-optische verschynselen in verband met het Hall-effect." (Verhand. der Kon. Akad. v. Wetensch. te Amsterdam, eerste sectie, Deel V., No. 3.) A translation of the original paper in French has since appeared (Archives néerlandaises (2) 1, p. 119, 1897).

[2] Goldhammer, Wied. Ann., 46, p. 71, 1892.

[3] Goldhammer, l. c., p. 76.

[4] Lorentz, Arch. néerl. 19; Versl. en Meded. Amst. (2) 19, p. 233, 1884. Van Loghem, Doctordissertation, Leyden, 1883.

[5] Wind, Versl. Kon. Akad. v. Wetensch. Amst., 3, p. 82, 1894; Verh. d. Physik. Gesellsch. Berlin, 13, p. 84, 1894.

the Hall-effect together with magneto-optic phenomena as far as observed (viz.: the Faraday magnetic rotation of plane of polarization and the Kerr-effect). This theory is to be considered as a transposition, and at the same time as an extension, of Lorentz's theory. The author was not only advised by Professor Lorentz himself to undertake this work; but the latter also favored him throughout his whole study on the subject with his kind instruction and his most valuable remarks and assistance.

The present paper also states the relation existing between the present theory and some of the older ones, especially those of Drude[1] and Goldhammer. It also shows that one may predict, apart from any special theory, some of the main features of reflection from magnetized mirrors by using only two general principles.

At the end of his paper the author derives from a peculiar conception as to motion of electricity in general the relation between electric current and electric force in a magnetic field, which initially had only been stated as a hypothesis.

§ 1. *Definitions and Notations.*

The diagram Fig. 1 shows the system of coördinates used.

For the sake of brevity the notation of vector algebra, as used by Lorentz[2] is used in this paper. A vector is represented either by a German letter simply, or by the symbols for the components in the direction of the coördinate axes, separated by commas and enclosed in parentheses. For example $\mathfrak{C}$ or (u, v, w), $\mathfrak{F}$ or (X, Y, Z), $\mathfrak{H}$ or (α, β, γ) representing electric current, electric force, and magnetic force respectively. The tensor of a vector will sometimes be denoted by the corresponding latin letter.

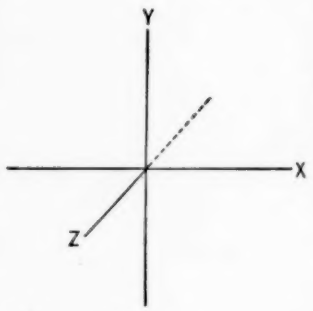

Fig. 1.

The symbol $\mathfrak{A}_x$ represents the component of the vector $\mathfrak{A}$ in the direction of the axis of x, $\mathfrak{A}_n$ its component nor-

[1] Drude, Wied. Ann. 46, p. 353, 1892.

[2] Lorentz, "Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern." Leyden, 1895.

mal to a definite surface, $\mathfrak{A}_h$ the component in a definite direction tangential to such a surface. At the interface of two media the symbols $(\mathfrak{A}_n)_1$ and $(\mathfrak{A}_n)_2$ are to represent the limiting values, to which the normal component $\mathfrak{A}_n$ of the vector $\mathfrak{A}$ approaches on both sides.

$\dot{\mathfrak{A}}_x$, $\ddot{\mathfrak{A}}_x$ etc., are to denote $\partial\mathfrak{A}_x/\partial t$, $\partial^2\mathfrak{A}_x/\partial t^2$ respectively, $\dot{\mathfrak{A}}$ is an abbreviation of $(\dot{\mathfrak{A}}_x, \dot{\mathfrak{A}}_y, \dot{\mathfrak{A}}_z)$, $\ddot{\mathfrak{A}}$ of $(\ddot{\mathfrak{A}}_x, \ddot{\mathfrak{A}}_y, \ddot{\mathfrak{A}}_z)$ and so on. The vector product of $\mathfrak{A}$ and $\mathfrak{B}$ has $[\mathfrak{A}.\mathfrak{B}]$ as its symbol and is the same as the vector $(\mathfrak{A}_y\mathfrak{B}_z - \mathfrak{A}_z\mathfrak{B}_y, \mathfrak{A}_z\mathfrak{B}_x - \mathfrak{A}_x\mathfrak{B}_z, \mathfrak{A}_x\mathfrak{B}_y - \mathfrak{A}_y\mathfrak{B}_x)$; the scalar product of the same vectors is designated by $\mathfrak{A}\mathfrak{B}$ and equals $AB\cos\theta$, θ representing the angle between the two vectors.

Rot $\mathfrak{A}$ stands for

$$\left(\frac{\partial\mathfrak{A}_z}{\partial y} - \frac{\partial\mathfrak{A}_y}{\partial z}, \frac{\partial\mathfrak{A}_x}{\partial z} - \frac{\partial\mathfrak{A}_z}{\partial x}, \frac{\partial\mathfrak{A}_y}{\partial x} - \frac{\partial\mathfrak{A}_x}{\partial y}\right),$$

Div $\mathfrak{A}$ for

$$\frac{\partial\mathfrak{A}_x}{\partial x} + \frac{\partial\mathfrak{A}_y}{\partial y} + \frac{\partial\mathfrak{A}_z}{\partial z}.$$

§ 2. *Disturbances periodical in time and space.*

In the present paper electromagnetic disturbances in isotropic media only are to be considered, and only such as, when not stationary, have a periodical character, so as to depend upon quantities which may be represented as functions P of the form

$$P = e^{-\frac{2\pi}{T}(d_2 x + e_2 z)}\left(A_1 \cos\frac{2\pi}{T}(t - d_1 x - e_1 z)\right.$$

$$\left. + A_2 \sin\frac{2\pi}{T}(t - d_1 x - e_1 z)\right). \qquad (4)^1$$

By introducing in the common way[2] complex magnitudes instead of real ones, we may also represent those quantities as functions P' of this form

$$P' = (A_1 + iA_2)e^{\delta[t-(d_1+id_2)x-(e_1+ie_2)z]}, \qquad (5)$$

where

[1] The numeration of the equations is in this extract the same as in the original paper.

[2] The original paper enters into more details as to this point and as to the nature of those disturbances which are characterized by the equations (4) and (5).

$$\delta = -\frac{2\pi i}{T}.$$

In these formulæ t is the time and T the period of the disturbance considered, the latter being supposed equal for the separate quantities characterizing the physical state of the medium, and of the same order of magnitude as the periods of vibration of light. The symbols d_1, d_2, e_1 and e_2 denote constants which have also a common value for all the above quantities considered, A_1 and A_2 to the contrary constants which in general are not the same for the different quantities. Putting $A_1 + A_2 = A$, $d_1 + id_2 = R \sin \varphi$ and $e_1 + ie_2 = R \cos \varphi$, we may write (5) in this form

$$P' = AP = Ae^{\delta[t - R(x \sin \phi + z \cos \phi)]}. \tag{6}$$

The ordinate y not appearing in the formulæ (4), (5) and (6) shows the axis of y to be parallel to the fronts of equal phase as well as to the fronts of equal intensity. In the case of an absorbing medium, where these fronts do not coincide so as to form a single wave front, the axis of y is therefore supposed perpendicular to the plane of incidence of the light considered.

As mentioned above we shall have to consider oscillations of extreme shortness only; hence we are allowed to disregard the magnetization of the medium arising from those magnetic forces which correspond to the periodical disturbances themselves. If, besides these periodical magnetic forces, there is in the field some constant magnetic force $\mathfrak{N}$, we shall have to attribute a certain influence to the magnetization of the medium corresponding to *this* force, but we do not need to assume any *variation* of magnetization to be caused by those periodical magnetic forces.

§ 3. *The fundamental equations, when no magnetic force* $\mathfrak{N}$ *is superposed.*

Denoting the electric current (in electromagnetic units) by the symbol $\mathfrak{C}$, we have, according to Maxwell, at any point in space

$$\text{Div } \mathfrak{C} = 0, \tag{I}$$

and at the interface of two media

$$(\mathfrak{C}_n)_1 = (\mathfrak{C}_n)_2. \tag{II}$$

For our case—where the magnetic properties of the medium are of no consequence—we may introduce the vector magnetic force $\mathfrak{H}$ as defined by the equations:

$$\text{Rot } \mathfrak{H} = 4\pi\mathfrak{C}, \tag{A}$$

$$\text{Div } \mathfrak{H} = 0, \tag{III}$$

$$(\mathfrak{H}_n)_1 = (\mathfrak{H}_n)_2, \tag{IV}$$

to which also may be added

$$(\mathfrak{H}_h)_1 = (\mathfrak{H}_h)_2. \tag{V}$$

The electric force $\mathfrak{F}$ satisfies the equations

$$\text{Rot } \mathfrak{F} = -\dot{\mathfrak{H}}, \tag{B}$$

$$(\mathfrak{F}_h)_1 = (\mathfrak{F}_h)_2, \tag{VI}$$

which we shall consider as one of the fundamental hypotheses of Maxwell's theory (according to Heaviside and Hertz), though there are several methods of deriving these equations by dynamical reasoning from other hypotheses, which may be more readily represented to our mind.

§ 4. *A farther relation between* $\mathfrak{C}$ *and* $\mathfrak{F}$.

The equations (A), (B), (I) to (VI) do not altogether suffice to offer a complete description of common electromagnetic phenomena. For this purpose there is still a farther relation between the vectors $\mathfrak{C}$ and $\mathfrak{F}$ wanted. It is shown by a simple course of reasoning that there are good grounds for assuming this relation to take the form

$$\mathfrak{C} = p\mathfrak{F}, \tag{37}$$

where p denotes a complex constant, depending upon the medium and the period T.

§ 5. *The fundamental equations, a constant magnetic force* $\mathfrak{N}$ *being superposed.*

The system of equations (A), (B), (37), (I) to (VI) contain the explanation of common optical phenomena, if light is supposed to consist of periodical electromagnetic disturbances of extremely

short periods. Moreover, the description is implied of such electric phenomena as occur in conductors carrying steady currents. Experience has shown, however, that these equations are not quite satisfactory either for optical phenomena or even for steady currents, if a medium is concerned in which in any manner whatever a constant magnetic force is maintained. Since neither the Hall-effect nor the Faraday and the Kerr-effect are predicted by the equations, the existence of these phenomena proves that the mutual relation between the electromagnetic vectors is not exactly expressed by these equations in the case of a magnetic field. Of course there are different methods of extending the system of equations to this more general case. It does not seem desirable, however, to try to accomplish this by modifying equations (I) and (II), these being the expressions of the fundamental property of the solenoidal distribution of electric current. The relations (A), (III) to (V) may be left altogether unchanged if we continue to consider them simply as determining the vector $\mathfrak{H}$. In doing so we should not consider this vector any more as any true magnetic force whatever, but only as that magnetic force which would correspond to the given distribution of current, the magnetic force $\mathfrak{R}$ being zero. So there remain only the equations (B), (VI) and (37) which we may attempt to modify so as to make the whole system of equations imply the new phenomena.

Some time since the researches of Hopkinson, Lorentz, and others showed that a mathematical description of the Hall phenomenon is obtained by assuming that the relation between the vector $\mathfrak{C}$ and $\mathfrak{F}$ in a magnetic field $\mathfrak{R}$ is represented by the formula

$$\mathfrak{F} = \frac{1}{p}\mathfrak{C} - q\,[\mathfrak{R} \cdot \mathfrak{C}],^{1} \qquad \text{(C)}$$

instead of (37), where q denotes a (*real*) constant, depending upon the properties of the medium concerned. If we examine whether the same extension is sufficient to account for the magneto-optical phenomena also (as would correspond to the original Lorentz

[1] Completely equivalent to (C) is the equation

$$\mathfrak{C} = p\mathfrak{F} + r[\mathfrak{R} \cdot \mathfrak{F}], \qquad \text{(C')}$$

when $r = p^2 q$, and powers of pqN are neglected as compared with pqN itself.

theory as elaborated by Van Loghem), we find that the modification referred to does not completely answer the purpose. In the following pages I shall try to show, however, that it is only necessary to assume the value of q to be in general *complex*, in order to have in the equation (C) a substitute for equation (37), thus raising the whole system of equations at once to the rank of a *mathematically complete description* of electromagnetic phenomena.

Of course this extension of Lorentz's hypothesis leaves his explanation or description of the Hall-effect quite undisturbed; the imaginary part of the constant q would show an influence upon this phenomenon, only if the latter were observed with rapidly alternating currents. As to a somewhat arbitrary interpretation of the assumption of a complex value for q, see paragraph 16.

§ 6. *Propagation of Light in a Medium which is magnetized in the Plane of Incidence.*

Our fundamental equations being given, we may at once pass to the solution of the following question: What are the constant factors which must appear before the factor of periodicity P in the expressions for the components of $\mathfrak{C}$, $\mathfrak{F}$ and $\mathfrak{H}$ in the case of a disturbance propagated through any medium, a magnetic force $\mathfrak{N}$ (parallel to the plane of incidence) being applied; and what is the condition to be satisfied by the constant R, upon which, according to (6), P is dependent?

By substituting the value of $\mathfrak{F}$ given by equation (C) in (B) and applying equation (I), (B) becomes

$$\frac{1}{p}\operatorname{Rot}\mathfrak{C} + qN\frac{\partial}{\partial z}\mathfrak{C} = -\dot{\mathfrak{H}}, \qquad (B_a)$$

if the axis of z be taken in the direction of $\mathfrak{N}$.

As a first approximation we shall everywhere neglect terms containing powers of pqN as compared with pqN itself.

All the quantities considered depending upon t and the coördinates by the factor P only, we may evidently write

$$\frac{\partial}{\partial t} = \delta., \quad \frac{\partial}{\partial x} = -\delta R\sin\varphi., \quad \frac{\partial}{\partial y} = 0., \quad \frac{\partial}{\partial z} = -\delta R\cos\varphi.$$

Equation (I) thus reduces to

$$u \sin \varphi + w \cos \varphi = 0, \tag{42}$$

and we may write, when putting a and b for two constants, which for the moment are left undetermined,

$$\left.\begin{aligned} u &= a \cos \varphi \,.\, \mathrm{P} \\ v &= b \,.\, \mathrm{P} \\ w &= -\, a \sin \varphi \,.\, \mathrm{P} \end{aligned}\right\} . \tag{43}$$

From (B_a) we find

$$\left.\begin{aligned} \alpha &= -\, R \cos \varphi \left[\frac{1}{p} b - qNa \cos \varphi \right] . \mathrm{P} \\ \beta &= R \left[\frac{1}{p} a + qNb \cos \varphi \right] . \mathrm{P} \\ \gamma &= R \sin \varphi \left[\frac{1}{p} b - qNa \cos \varphi \right] . \mathrm{P} \end{aligned}\right\} , \tag{44}$$

and (A) furnishes, besides an identity, the following conditions which must be satisfied by a, b and R:

$$b = \pm ia, \tag{45}$$

$$R^2 = \frac{4\pi p}{\delta(1 \pm pqNi \cos \varphi)}. \tag{46}$$

If we denote by R_0 the value which R takes when $\mathfrak{N}$ is zero, we have

$$R_0^2 = \frac{4\pi p}{\delta}, \tag{D}$$

and (46) becomes, by substituting

$$\tfrac{1}{2} pqN = \mu \tag{E}$$

and neglecting powers of μ,

$$R = R_0(1 \mp \mu i \cos \varphi). \tag{F}$$

Finally, substituting (45), (E) and (F) into (43), (C) and (44), we obtain

$$\left.\begin{aligned} u &= a\cos\varphi \,.\, \mathrm{P} \\ v &= \pm ia \,.\, \mathrm{P} \\ w &= -a\sin\varphi \,.\, \mathrm{P} \end{aligned}\right\}, \text{(G)} \qquad \left.\begin{aligned} X &= \frac{a}{p}(\cos\varphi \pm 2\mu i) \,.\, \mathrm{P} \\ Y &= \pm i\frac{a}{p}(1 \pm 2\mu i\cos\varphi) \,.\, \mathrm{P} \\ Z &= -\frac{a}{p}\sin\varphi \,.\, \mathrm{P} \end{aligned}\right\}, \text{(H)}$$

$$\left.\begin{aligned} \alpha &= \mp iR_0\frac{a}{p}\cos\varphi(1 \pm \mu i\cos\varphi) \,.\, \mathrm{P} \\ \beta &= R_0\frac{a}{p}(1 \pm \mu i\cos\varphi) \,.\, \mathrm{P} \\ \gamma &= \pm iR_0\frac{a}{p}\sin\varphi(1 \pm \mu i\cos\varphi) \,.\, \mathrm{P} \end{aligned}\right\}. \qquad \text{(K)}$$

In the equations (G), (H), and (K), together with (D), (E), and (F) and the expression for P, we find a complete mathematical description of the electromagnetic disturbances of the period T which are consistent with our theory in the medium considered. Their dependance upon the constants p and q of the medium, upon N, which determines the magnetization of the latter, and finally upon the angle φ, is clearly indicated by these equations. They represent two different solutions, according as the upper or lower of the double signs $\pm$ and $\mp$ is used. From (F) we see that the two rays of light corresponding to these solutions have their velocities of propagation and coefficients of absorption slightly different ; as for the rest they may be considered as circularly polarized rays, one right-handed, the other left-handed.

(*To be concluded.*)

MINOR CONTRIBUTIONS.

ON A METHOD OF OBTAINING PORES OR CAPILLARY CANALS OF SPECIFIED DIAMETER.

C. BARUS.

IN the treatment of small particles of suspended solid matter, I have often found myself at a loss for some method to specify the diameter of such particles directly, or at least without the aid of some complicated and possibly visionary hydrodynamic hypothesis. The method proposed elsewhere[1] consists in providing a suitable strainer for the purpose. To measure the diameter of the pores, I determined the pressure necessary to force air through the capillary orifices, when the body is saturated with and submerged in water. In this case $2\pi r T' \cos a = \pi r^2 \cdot p \times 10^6$ nearly, where r is the mean radius of the canals, p the pressure of the gas in atmosphere, T' and a the surface tension and angle of capillarity (practically zero) for water, all data except p being referred to dynes and centimeters. Some few experiments made with a sample of porcelain showed that as much as eight atmospheres were required to initiate capillary transpiration of the gas through the wet porcelain septum, dislodging the water on the way. Hence, from the equation just stated, $r = 18 \times 10^6$ cm., or the capillary pores were of a diameter even below the wave-length of violet light. Colloidal solutions (so far as examined) passed easily through this septum whereas precipitates (even of colloids) usually failed to do so. I then showed how it would be possible to further decrease the given diameter of pore by vitrifying the septum more and more perfectly, and suggested this method for the treatment of the question of colloids. My present purpose is the reverse of this, as I am endeavoring by a special method to secure pores increasing in size from a given minimum. Said pores would be more serviceable for the treatment of ordinary precipitates than the much finer porcelain pores.

The essential part of my apparatus is a length of elastic rubber tubing, the thickness of the walls of which is chosen larger or smaller ($\frac{1}{32}''$ to $\frac{1}{8}''$), according as finer or coarser pores are wanted. This tubing is punctured radially with holes, disposed as nearly together as possible, from end to end of the rubber tube. A specially constructed comb containing about 50 equidistant cambric needles lying in the same plane

[1] Barus, American Journ. (3), xlviii, p. 451, 1894.

was used for puncturing. This was actuated by a mechanism similar to a screw copying press and I was thus enabled to pierce from 5,000 to 10,000 holes per linear meter through the walls of the tube in question. A wooden core was inserted into the inside of the tube during punching to make sure that each needle penetrated the substance. I have since devised an automatic mechanism for the same purpose, but the description of these and similar processes is beyond my purpose here.

The interesting feature of the tube so prepared is the fact that, after the needles are withdrawn, the holes close almost completely and their position is all but invisible, to the unaided eye. Furthermore, the tube made radially porous in this way, is a *valve* of a peculiar kind, possessing the important property that the pores can be distended or brought to vanish in any degree at will.

Let *A*, in figure 1, be a cross section and *BC* a longitudinal section of the tube. Then if there is excess of pressure on the inside, the tube will

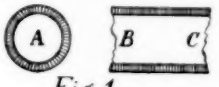

Fig. 1.

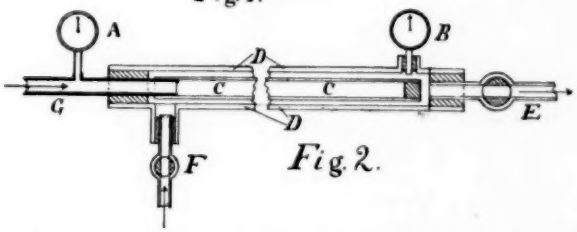

Fig. 2.

enlarge and the pores open proportionately. If there is excess of pressure on the outside of the tube (with proper precautions against collapsing), it will be radially compressed and the pores closed. Thus it is merely necessary to regulate the internal and external pressure, to provide a mean size of pore of any diameter (within reasonable limits). Moreover a liquid or gas can pass by its own pressure from within outwards; but no fluid can under the same conditions pass from without inwards. The tube might be called semipermeable if the word had not been already coined in a different sense.

To use this device for obtaining pores of specified diameter, the following apparatus (to which a great variety of forms either horizontal or vertical may be given) is available. The punctured elastic tube *C* is axially surrounded by a cylindrical wide glass tube *DD*. Its ends are provided with tubulures *F* for water, *G* for gas (or the turbid liquor to be strained) and *E* for the efflux. All tubes contain suitable stop cocks and two gauges *A* and *B* register the pressure on the two sides of the porous tube *C*.

To calibrate this tube it is necessary to allow gas to enter at *G* under

pressure, and water similarly at F, and to observe the pressure excess of the gas gauge A over the liquid gauge B when bubbles first begin to appear. For at this point, if the gas pressure be gradually increasing, the capillary pressure or resistance has just been overcome by the gas pressure within the tube. Beyond this, if the latter is further increased, the bubbles escape at first as a fine spray and finally tubulently showing that the holes are more and more open. In case of each tube, however, a definite initial pressure is necessary to admit of the discharge of the contents of gas.

To give examples: using elastic tubing $\frac{1}{2}''$ external and $\frac{1}{8}''$ internal diameter, the pressure necessary to open the walls when the punctures were made with needles usually exceeded an atmosphere. Hence, the initial pores under these conditions were of radius $r = 142/10^6$ cm., or somewhat less than 10 times as large as in the above experiment with porcelain. With thin walled tubes the size of the holes may be increased indefinitely. Thus in the case of a tube $\frac{1}{2}''$ wide with walls $\frac{1}{32}''$ thick of red rubber, the pressure to open the tubes could be reduced to $\frac{1}{10}$ atm., showing $r = .0014$ cm. initially.

Naturally, much depends on the size of the punctures made or the size of the needles in the comb. It is to be noted moreover that in all these cases there is effective pressure on both sides of the walls of the tube. Hence the size of the hole is conditioned by the absolute pressure values as well as by the pressure excess, or r depends on the pressure sum as well as on the difference. To obtain very fine holes (apart from puncturing) one would therefore make use of large pressures. If for pressures not too high a tube is bent, bubbles appear sooner on the convex side than on the concave side of the flexure.

It is interesting to note that if the inner contents of the tube is gaseous, the latter remains quite dry within under all ordinary circumstances, proving that the valve action mentioned above is very perfect and that the pores increase in diameter from zero. Again, if the tube, even with thin walls, be filled with water and hung up closed at both ends in the air, the water will gradually evaporate as through unglazed porcelain and the hose thereafter partially collapse, owing to the external pressure of the atmosphere.

Incidentally, I may point out that if a tube of baked (unglazed) porcelian be filled with water and sealed, and if the tube be kept moist as by shaking, that a vacuum quite up to the vapor tension of water should eventually be produced within by the evaporation of the water on the outside of the porous tube. One ought, therefore, to be able to lift water to a height of say 30 feet, indefinitely in this way, supposing the tube to be perfect and to communicate at its lower end with a cistern. A single imperfect pore ($r > .0001$ cm. say), or a small fissure, or the like would allow the ingress of air and impair the lift.

BROWN UNIVERSITY, PROVIDENCE, R. I.

On the Methods of Measuring Mean Horizontal Candle-Power.

C. P. Matthews.

THE photometrist has at his command two methods of obtaining the mean horizontal intensity of a glow lamp. The first, and that in general use, consists in finding the mean radius of the candle-power distribution curve, as taken point by point; the second consists in whirling the lamp about a vertical axis of symmetry at a speed such as to eliminate flickering, and in making photometer settings with respect to the mean illumination as perceived on the photometer disk. This latter method, it appears, was first snggested by Professor C. R. Cross.[1]

Considerations based on Helmholtz's law[2] for the action of intermittent light upon the retina would indicate that the two methods should give the same result. The writer has held the opinion, however, that it would be well to put the psychological point involved to actual test under the unique conditions met in a spinning glow-lamp, particularly so since the Committee on Light Standards of the American Institute of Electrical Engineers has recently recommended the general adoption of the whirling method. In this opinion he has been joined by several authorities on physiological optics, with whom he has communicated.

Measurements made with Mr. C. H. Sharp about a year ago indicated that any existing difference was at any rate small, and that much care would be needed to bring it out. With a view of making the matter definitive the writer has recently taken new data, with all possible care, the results of which are given below.

But before passing to these results, it may be well to add that there are, aside from any possible difference of the character mentioned, ulterior effects which might put the results obtained by the two methods at variance. Thus Professor Thomas[3] has pointed out the existence of a "phantom filament." The light reflected from the bulb of a glow-lamp produces marked, sudden variations in candle-power as the lamp is slowly turned on its vertical axis. When the mean horizontal candle-power is obtained from readings taken at 30-degree intervals, these variations are likely to be excluded; but when the lamp is rapidly rotating they necessarily contribute to the mean as perceived by the eye. Furthermore, there is always in lamps with slender filaments a certain amount of filament distortion due to the speed of rotation. This distortion is marked in lamps

[1] Franklin Institute Congress in 1884.

[2] Physiologische Optik, 2d ed., p. 483.

[3] Proceedings of the Electrical Congress, Chicago, 1893.

having filaments of the double-loop type. The planes of the loops tend to separate as the speed of rotation increases. If the filament is anchored the motion is confined to the top of the loops. In any case there is likely to be an initial lack of symmetry in the filament which, if the speed be sufficiently increased, will cause it to bend until it touches and ruptures the bulb. Now, since the fundamental law of photometry is not one of simple distances, any bending, even though symmeurical, must change the effective center of the light source. This effect, although of trivial importance when a long bar is used, must yet be regarded as real.

The incandescent lamp as manufactured to-day gives a horizontal distribution that is nearly circular and hence is ill adapted to test the point at issue. An irregular distribution was obtained for the following measurements by pasting pieces of black paper on the bulb of the lamp. As some distortion occurred at high speeds, the top of the lamp was entirely screened by a hood of black paper, so as to effectually cover the points at which the bending was noticeable. Finally the lamp was whirled at different speeds, from a minimum which caused a barely perceptible flicker to a maximum beyond which it was not considered safe to go. No variation in intensity with speed was found.

The distribution curve shown in Fig. 1 was then taken, the readings being made at 10-degree intervals. The mean bar readings and the corresponding light ratios, multiplied by 10, appear in the sub-joined table. A 360-part bar was used and a standard of approximately 10 candle-power, purposely adjusted to give about the same intensity and color as the lamp under test.

Degrees.	Mean reading.	Light-ratio × 10.	Degrees.	Mean reading.	Light-ratio × 10.
0	183.1	10.71	190	182.2	10.50
10	185.7	11.34	200	182.2	10.50
20	184.8	11.13	210	178.1	9.58
30	173.8	8.70	220	174.3	8.80
40	160.5	6.48	230	204.5	5.78
50	159.4	6.32	240	162.35	6.74
60	169.9	7.99	250	153.55	5.52
70	160	6.4	260	153.75	5.55
80	165.6	7.26	270	172.1	8.39
90	188.1	11.97	280	180.9	10.02
100	187.8	11.88	290	182.5	10.56
110	187.7	11.86	300	181.6	10.36
120	187.15	11.73	310	181.85	10.40
130	187.55	11.83	320	182.5	10.56
140	184.4	11.03	330	185.9	11.40
150	185.9	11.4	340	183.55	10.82
160	183.45	10.79	350	182.2	10.50
170	181.75	10.38	360	183.5	10.80
180	180.7	10.16			

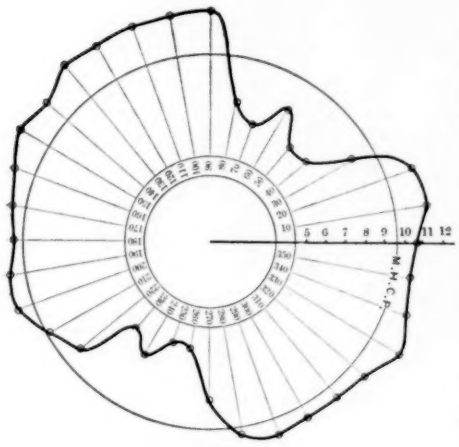

Fig. 1.

To facilitate integration the readings were also plotted in rectangular coördinates. The planimeter shows from this latter curve a mean horizontal candle-power of 9.655. The mean of all readings on the whirling lamp is 9.649. The difference, amounting to .006 or about .06 per cent. is entirely with the limits of photometric error.

The obvious conclusion is, then, that the two methods lead to the same result *when there is no color difference.* Ferry found that the error resulting from the use of the sectored disk was not a function of the luminosity ratio of the sources compared but was dependent on their color. It would be natural to expect the occurrence of similar phenomena in the present instance. The writer hopes to report on this later.

[1] PHYSICAL REVIEW, Vol. I., p. 338.

Messrs. J. Walter Esterline and R. W. Treat rendered most valuable aid in taking the foregoing measurements.

ELECTRICAL LABORATORY, PURDUE UNIVERSITY,
July, 1897.

NEW BOOKS.

Die Photographie der Gestirne. By DR. J. SCHEINER, Leipzig. Wilhelm Engelmann, 1897. 8vo, 338 pp. With an Atlas of 11 plates, heliogravure.

Like the "Spectrum Analysis of the Stars," by the same author, this volume may serve as a hand-book for astronomers, as well as a text-book for students. The dominant idea, throughout, is to present celestial photography as an exact science, with claims beyond the mere reproduction of pictures.

The object of photography is treated as two-fold, the representation of the heavens and the measurement of specific subjects. It is needless to say, that the first includes the bulk of accomplishment, hitherto. The volume in question will certainly be an aid, in advancing the second.

Knowledge of the practical details of the photographer's art, is taken for granted; and the treatment of the special object in view begins with the discussion of the effect of light waves of different lengths, upon the plate. Some account, historical and technical is, however, given of the various processes; daguerreotype, and wet and dry plates, with the results which have been reached by their use. The first named process is described, as being an ideal one for astronomical photography, in many respects, if it were possible to increase the sensitiveness, to the required degree.

The class and size of instrument, with the proportion of objective to focal length, are thoroughly discussed, with reference to the various purposes; the production of flat pictures, for detail in bright objects, and for definition in star images. The losses by reflection, and absorption, are considered in the case of refractors; with the loss by atmospheric absorption, greater in photographic work than in visual observations.

With respect to the Bruce telescope, of the Harvard College Observatory, it is questionable whether the direct denial of a statement, made by another—in a popular article—is not somewhat out of place in a book of this permanent character. The periodicals are open to such discussions. In these, controversy is often carried to excess; and the main points of contention are lost sight of, in the cross fire of opinions, not of results, but of their interpretation.

Reflectors are recommended, especially, for photography, where great light strength is desired, and where sharpness is not essential.

The difficulties, in the way of exact measurement of photographic

plates, are treated at length; distortion, the effect of refraction—different in degree from that of visual observations—unequal adjustment of the plates with respect to the optical axis, change in focus, and, finally, bad definition. Unsteadiness of the air is described, as of three classes; though the author evidently falls in with the practice of most observers, in adopting a numerical scale, with which to express the conditions of the night.

The reduction methods, by both rectilinear and polar coördinates, are described, together with the instruments employed at Leiden, Potsdam and elsewhere. Kapetyn's method, for the measurement of the *Durchmusterung* plates of the Cape Observatory, is given with much detail.

In referring, later on, to this undertaking, as affording astronomers a foundation, similar to the visual work of Argelander and Schönfeld, the author appears to have lost sight of the fact that the visual D. M. has already been considerably extended beyond their limits. And, that it is several years, since the observations were reduced and published, catalogue and maps.

In Photographic Photometry, Part II., the difficulties in adjusting a scale of photographic magnitudes, to that of direct observation, is considered. The adoption of a different scale may eventually be decided upon, and the stars would then require grouping according to their classes of spectra. Besides the obstacles, in the way of determining magnitudes, due to color, when two plates are to be compared or joined together, there are others scarcely less serious. There is not only the lineal relation between the size of the disc, and the magnitude, to be considered; but also the more complicated relation, possibly logarithmic, between the disc and the exposure time; besides the uncertainty due to fluctuating conditions, and to the effect of atmospheric absorption. This last is well illustrated, in the difference between two treatments of the effect, those of Müller and of Schaeberle. Differential methods can, however, be used, and each plate can be treated by itself, with a scale to be adopted from the stars of known magnitude, in the region included.

With respect to the exposure time, Dr. Scheiner concludes from his own experience that there is a gain of but ½ to ¾ mag. for 2½ times the exposure, in place of one whole magnitude, generally used as the first approximation. If the smallest value is correct, with the standard 13-inch telescope, adopted for the international scheme of mapping the sky, 6 minutes would be required for 11 mag. and 4 hours for 13 mag. in place of 2 minutes, and 10 minutes, respectively. There is great divergence of authority.

Late reports of the progress, made by the various observatories that entered this scheme, show most unequal fulfilment; execution has often fallen far behind promise, in the photographic field.

In the historical chapters, Part III., full credit is given to Bond; not only for his early tests of the photographic method, but as well for his far-sighted prediction of the possible fruits, and the limitations of the method. In fact, throughout the work, we find the credit given to original investigations, and to the accomplishment of results.

Dr. Gould's photopraphs of Southern clusters, whose early appearance is referred to in anticipation, have since been published. These are among the substantial fruits of astronomical photography, as an exact science.

Note is also made of some of the fine results, which have been attained in measures of parallax, by plates. It will be, perhaps, in special researches of this nature that efficient work is to be anticipated.

In the chapter devoted to the Sun, a full account is given of all of the Transit of Venus expeditions of 1874, and the results. The following transit, of 1882, is less extensively treated. Other chapters are given to the Moon, the planets, and to comets, clusters and nebulæ.

In the Bibliography, are references to original papers, and in some cases, to the discussion or compilation of the work of others.

The Atlas contains some of the most notable pictures, that have been obtained by photography; showing, in several cases, details quite beyond the power of any telescope, visually. A few plates are marred by the photographer's name, inscribed across the face.

In striking again the keynote of his subject, Dr. Scheiner writes "that it will not be long before the present relations will be reversed, and it will be the exception to undertake an investigation by direct observation." This is perhaps going rather far; the human senses are delicate appliances, and the trained faculties and intelligence of the observer are not likely to ever pass entirely into disuse.

R. H. TUCKER.

Die Photometrie der Gestirne. DR. G. MÜLLER. Leipzig, Wilhelm Engelmann, 1897. pp. 510.

Up to the present there has been no complete treatise upon astronomical photometry, the oldest of the astrophysical researches. This volume furnishes a comprehensive review of the subject.

Beginning with the development of the theoretical laws, the history of the various methods which have been devised, for measuring the intensity of the light of the heavenly bodies, is given in full, with detailed description of the different forms of apparatus. The general results, which have thus far been obtained, follow in order.

The measurement of the intensity of a single source of light is first considered; then of a lighted surface. A surface reflecting light offers a more complicated problem; and, from the laws governing this case, the

albedo can be determined. As a practical application, the diameter of a body too small to measure directly, as is the case with most of the asteroids, can be obtained by comparison of its albedo and brightness. The albedo of the earth has been derived from measurement of the earth-shine upon the moon. The extinction of light, in the earth's atmosphere, plays an important part; and the development of the theory, based mostly upon empirical laws, is followed from the time of Bougner and Lambert, the founders of scientific photometry, down to the latest determinations made upon the summit of Säntis, at 8,000 feet altitude.

Both methods of comparison, by extinction, and by means of the equalization of light, are treated in connection with the apparatus employed. The first is to be recommended, for simplicity; while the latter is capable of greater precision. The results are obtained by varying the aperture, by absorbing mediums, by reflection with mirrors, by means of a rotating disc, and finally by polarization and reflection. All have been more or less successfully employed, and the merits of each method receive consideration. Zöller's photometer, the Pritchard-Kill photometer, and the meridian photometer of Pickering represent the highest efficiency, which has been reached by the use of instruments.

In considering the scheme of astronomical photometry, in its broadest sense, one is reminded forcibly of the extent to which our knowledge of the light of the fixed stars has been derived from direct estimates. These have been made either by the naked eye, for the various Uranometries; or by telescopic aid, for the regions covered by the several parts of the Durchmusterung.

The exactness of the scale of the Uranometer of Dr. Gould, is hardly surpassed by photometric measures. The relative brightness of the stars, in the Durchmusterung, can be relied upon within small limits of error; based, as they are, upon estimates which have amounted to several hundred thousand for each of the observers that have taken part in the scheme, at various epochs. The ordinary magnitudes of our star catalogues must fall far behind in exactness, since there is never the same continuity in the comparison; and since the precision of the position is the main object in view.

Photometric apparatus has been mainly devoted to the bright stars. The Oxford photometric Uranometry; and the Potsdam photometric D. M., of which the first volume has been issued, may be considered as supplementing and confirming the results of the direct estimates. Including the extensive photometric measures of the Harvard College observatory, it is probable that by the end of the present century all stars above sixth magnitude will have been determined will all desired accuracy.

Our knowledge of the variable stars has thus far been due, nearly entirely, to the direct comparison with other stars. The results gained by

the method of steps, first recommended by Argelander, fall but little below the best photometric measures.

For the sun, moon and planets, photometric instruments are, in general, necessarily employed. The results of determinations of light and albedo, for these objects, are reviewed in this volume. The history is followed from Herschel's first photometric work, up to the most refined and latest measures, at Potsdam.

Upon comets, there is but little to record. Of the Nebulæ, there is an interesting history of three which have been found to be variable.

Pickering's classification of the variable stars has been adopted; and the results are brought up to 1896 from Chandler's latest catalogue.

Among methods of photometry, which may be classed as being independent of direct measure by the eye, are those by use of actinometers, radiometers, and finally by the photographic process. Several schemes, for obtaining standards of comparison, are noted. One by exposing half the plate upon known stars; another by exposing upon a star, of which the intensity is varied, to produce a standard scale.

But, in summing up the results of photographic photometry, the author frankly admits that the great expectations of the last ten years have not been fulfilled. Though the precision that was anticipated may not be attained, the prospect of gathering, in a relatively short time, a great amount of material, to be worked up at leisure, is still a tempting one. That the optical method will ever be superseded by the photographic, is not, in the author's opinion, probable.

Tables of the reduction for phase angle, in the case of the planets, and for the extinction of light, are given in the appendix. And a list of references to contributions upon subjects relating to photometry.

In the body of the work, reference is made to original authorities: and credit given to all shades of opinion. In the effort to be impartial, one side of the question (with respect to photographing variables) is represented by a man distinguished by his researches in these lines; while opposed, is the opinion of a graduate student, making his first essays. It is, however, not always possible to sift out the immature work, which easily finds its way to print.

There is still need of much careful photometric work upon the planets; while the fixed stars offer a wide field for long continued observation.

R. H. TUCKER.

Introductory Course in Differential Equations. By D. A. MURRAY. pp. viii + 232. New York, Longmans, Green & Co., 1897.

The aim of this book is "to give a brief exposition of the devices employed in solving differential equations," and the design is to supply the needs of two classes of students: first, of students "in physics and en-

gineering who wish to use the subject as a tool;" second, of students "in the general courses in arts and sciences, * * * some of whom may be intending to proceed to the study of the higher mathematics."

There are successively treated

Chap. I, Formation of a Differential Equation;
Chap. II, Equations of the First Order and First Degree;
Chap. III, Equations of the First Order but not of the First Degree;
Chap. IV, Singular Solutions;
Chap. V, Applications to Geometry, Mechanics and Physics;
Chap. VI, Linear Equations with Constant Coefficients;
Chap. VII, Linear Equations with Variable Coefficients;
Chap. VIII, Exact Differential Equations and Equations of Particular Forms, Integration in Series;
Chap. IX, Equations of the Second Order;
Chap. X, Geometrical, Mechanical and Physical Applications;
Chap. XI, Ordinary Differential Equations with more than two variables;
Chap. XII, Partial differential Equations. Partial Differential Equations of the First Order. Partial Differential Equations of the Second and Higher Orders.

In the effort to make the course a brief one and to cover at the same time so much ground, much has been inadequately treated. Thus in Art. 15 the usual proof is given that the number of integrating factors of $Mdx + Ndy = 0$ is indefinitely great, the proof being based upon the existence of *one* such factor. But nowhere does the author prove or state the truth of the last assumption. In Chap. VI. is introduced in a rather summary way the symbolic notation $f(D)\,y = 0$, $D = \frac{d}{dx}$, — summary, because here the student is initiated into the calculus of operations, and for the first time, probably is dealing with a non-scalar algebra, No mention is made of the associative law, for example. The same incompleteness is to be remarked in Chap. XII., in treating of Partial Differential Equations with Constant Coefficients. Art. 118 reads as if a two-dimensional manifoldness always existed as integral of $Pdx + Qdy + R.\,dz = 0$! A little more space should have been devoted to Integration in series, Chap. VIII., at least a little fuller treatment more sufficient unto the needs of the first class of students mentioned above.

The chapter on Singular Solutions is poor, and one may doubt if a student will be able to form much of an idea as to what it is all about. After defining the discriminant manifoldness, the author goes on to say that the envelope of a singly infinite system of algebraic curves, $\varphi(x, y; c) = 0$, is the locus of the "ultimate points," *i. e.*, of the points of intersection of each curve with its consecutive. To be accurate, these points should be defined as the limit of the points of intersection, etc.

Then we read: "the envelope is part of the locus of the c discriminant relation." Of course, as above defined, the envelope is the *entire* discriminant manifoldness, and thus the author gets into trouble in discussing the nodal and cuspidal loci. In Art. 37 is stated, correctly enough, that when each curve of the system $\varphi(x, y; c) = 0$ has a double point or cusp, the latter is an "ultimate point." But then, according to definition, the nodal and cuspidal loci are part of the envelope, and this has been defined in Art. 33 as the Singular Solution! A teacher will be needed to carry the learner safely through Chap. IV.

In the other chapters Dr. Murray has done the work well, and the student will be able to follow the development easily and without being led into erroneous inferences. The last should be, after all, the aim of a text book, viz., that while an elementary treatment will not permit *fin de siècle* rigor, any leading of the student into error must be avoided. And so Dr. Murray starts out in Chap. II., with a timely reminder of the assumptions upon which the fabric rests.

The occasional summary of results, as on pp. 38, 118, 119, will be very handy. Numerous exercises are given, many relating to physical problems. But a feature to be highly praised is the introduction of numerous notes, historical, biographical and miscellaneous, the last covering pp. 189–207, and including proofs of the existence theorem for ordinary differential equations, the theorem on the number of arbitrary constants in the complete primitive of a differential equation of nth order, criteria for the independence of these constants, etc. A list of works on Differential Equations ends the book.

The printing is well done, and few typographical errors have been made. In post graduate work, and whenever the larger treatises are impracticable, the book will certainly be much used.

PERCY F. SMITH.